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RESEARCH REPORT No. EM-153

An Accurate Boundary Condition to Replace Transition Conditions at Dielectric-Dielectric Interfaces

JULIUS KANE and SAMUEL N. KARP

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Abstract

Most problems involving dielectrics cannot be solved exactly, e.g., diffraction by a dielectric wedge. It is the purpose of this work to describe a technique by which the transition conditions at an interface between two dielectrics can be conveniently and accurately replaced by a single mixed boundary condition. The method is useful when one is interested in the field in only one of the dielectrics. The boundary condition takes the form

$$\frac{\partial u}{\partial \nu} + Au + B\,\frac{\partial^2 u}{\partial s^2} = 0$$

where ν and s are the surface normal and tangent respectively. When a single interface is present, so that there is no diffraction, it is proved that the maximum percentage error, over the entire far field of an arbitrarily placed line source is small. For example, we can tabulate the maximum percentage error for transverse electric excitation as

Index of refraction = n	1.2	1.4	1.6	1.8	2.0
Maximum pct. error in far field	4.5	1.1	0.4	0.2	0.1

The percentage error becomes very large for $n \sim 1$ because for $n = 1$ there is no reflection at the interface; however the absolute error remains small and bounded.

Additional confidence in the technique is acquired by using the boundary condition to replace a dielectric interface in a non-trivial problem which does involve diffraction. A comparison of the results of the approximate version and the exactly formulated problem shows that there is excellent agreement between the two.

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Introduction

Most electromagnetic problems involving more than one dielectric medium are not capable of exact solution. An example of such a problem is sketched in Figure 1. One is interested in obtaining a solution of the wave equations

$$(\nabla^2 + k_i^2)\, u(x,y) = 0, \qquad i = 1, 2, 3$$

for the indicated geometry when a plane wave is incident in medium I. It is assumed that the area of prime interest is region I. The boundary conditions to be satisfied are continuity conditions determined from Maxwell's equations. As the problem stands, it is not solvable by available mathematical techniques. However, one notes that the two-part boundary-value problem of Figure 2 is amenable to the Wiener-Hopf procedure if BC_{12} and BC_{13} are mixed boundary conditions with constant coefficients.

The following procedure then suggests itself for obtaining an approximate field in region I:

(1) Replace the dielectric interface between regions I and II by a linear boundary condition $BC_{12} = C$, and do the same for the pair of regions I and III.

(2) Solve the problem of Figure 2 exactly to obtain an approximate solution to the problem of Figure 1 in region I.

The success of this approach depends upon constructing a boundary condition to replace a dielectric interface and validating the procedure.

In the present report (Part I of two parts) we discuss the formulation and justification of the new boundary condition, and in a subsequent report (Part II) we solve the problem of Figure 1 as an illustration of its use.

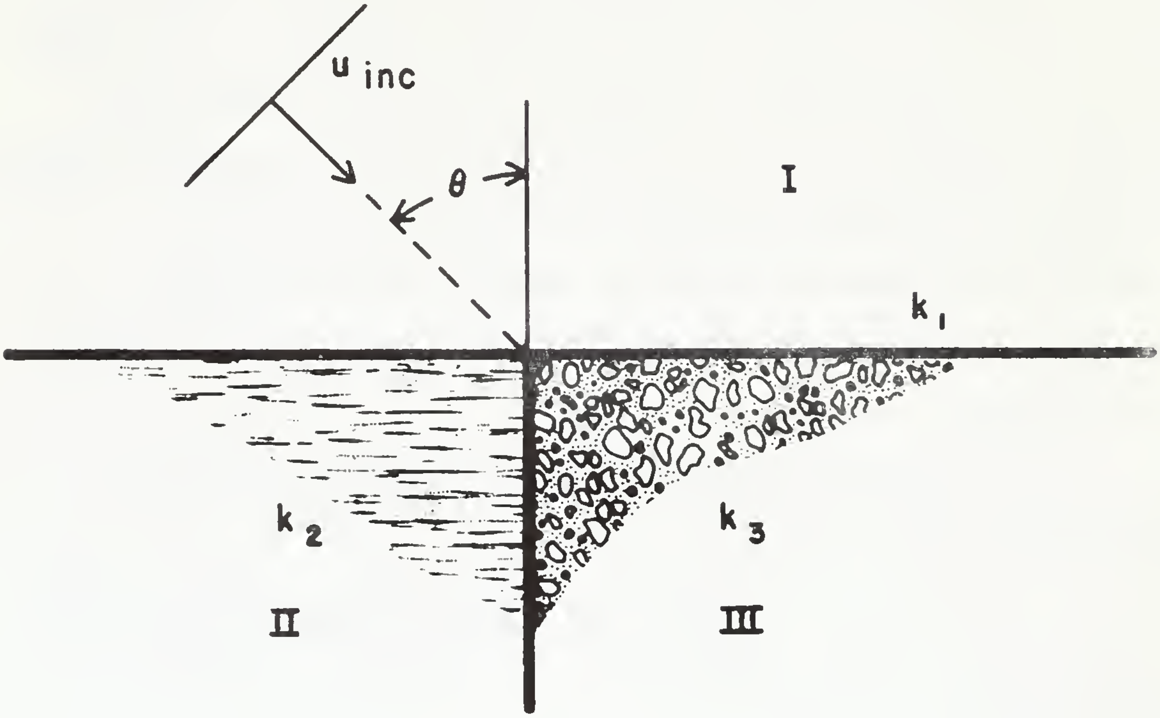


Figure 1: Idealized Three-Media Problem

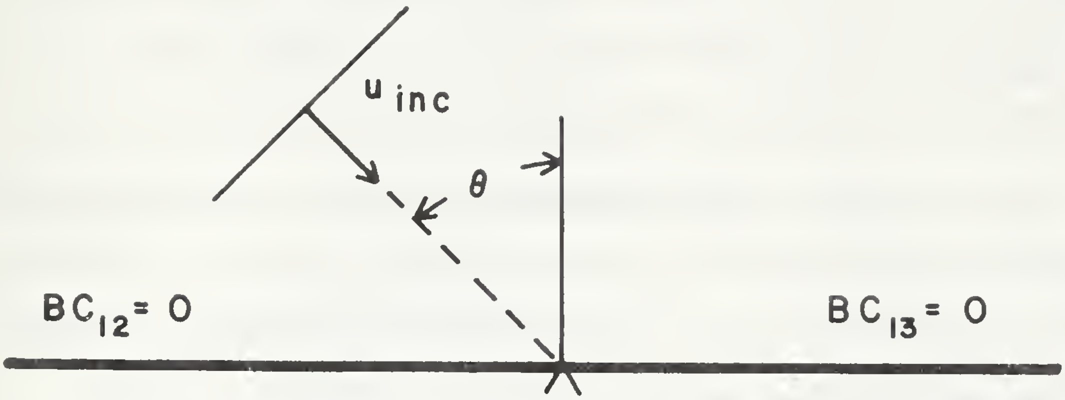


Figure 2: Reformulated Approximate Version of the Problem Shown in Figure 1 in Region I.

In section 1 we determine the coefficients in the new boundary condition by comparing the reflection coefficient it defines to the actual Fresnel reflection coefficient appropriate to a dielectric surface. We present numerical data that shows that there is little error in approximating the Fresnel coefficient if our methods are employed. Certain limitations and remarks concerning this procedure are presented in Section 2.

Sections 3 and 4 are devoted to an investigation of the utilization of the new boundary condition at the interface of a dielectric half-space. In these sections it is shown that the introduction of the boundary condition leads to a problem with a unique solution which satisfies reciprocity. Once reciprocity, uniqueness and existence have been demonstrated we are in a position to assert that this boundary condition leads to (1) well-defined problems, and (2) guarantees that the error in the far-field of the approximate version of the standard problem of the radiating line source above a dielectric interface is small.

Since the problem treated in sections 3 and 4 involves no diffraction, confidence in the utilization of the new boundary condition is amplified in Section 5, in which we compare an exact solution of a diffraction problem involving two dielectrics and its approximate counterpart. We find strikingly good agreement between the two results.

1. Construction of the Boundary Condition

In order to replace a dielectric interface by a linear boundary condition it is necessary to determine what characterizes the interface so far as wave propagation is concerned. If one solves the two-medium problem of a line source above a dielectric interface (Fig. 3) then one obtains integrals of the form (cf. Bremmer, [8]).

$$\int_C R(\theta)\, e^{ik\rho\cos\theta}\, d\theta \quad , \tag{1}$$

in addition to the field that would exist in the absence of the boundary. In the integral of (1), $R(\theta)$ is the appropriate Fresnel reflection coefficient and the contour C is the familiar contour defining the Hankel function.

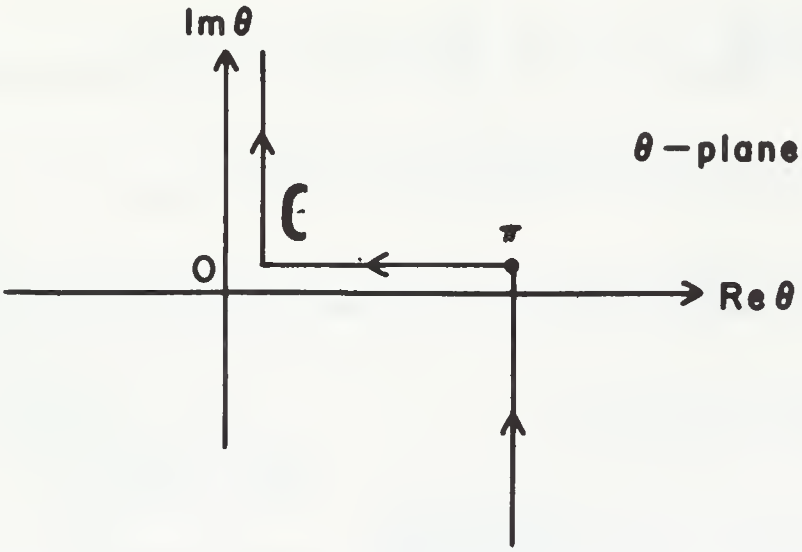


Figure 3: Contour defining the Hankel function $H_0^{(1)}(k\rho)$.

The diffracted field represented by (1) then has the interpretation of a summation of plane waves traveling in all directions, real and imaginary, which have as a weight factor the Fresnel coefficient defined in the complex θ-plane.

One can also observe that a linear boundary condition at the interface will define a reflection coefficient if we seek plane wave solutions. When medium (2) is almost conducting, but only then, it is, in fact, known that a boundary condition of the form[10]

$$u_y + \lambda u = 0$$

represents the phenomena fairly well. Therefore, for a more general case we seek to include more parameters so as to improve the approximation. For example, notice that the boundary condition

$$\left(\frac{1}{ik} \frac{\partial}{\partial y} + A + \frac{B}{k^2} \frac{\partial^2}{\partial x^2} \right) u = 0 , \qquad y = 0 \tag{2}$$

will generate for the wave equation

$$(\nabla^2 + k^2)\, u(x,y) = 0 , \qquad y \geq 0 \tag{3}$$

the reflection coefficient

$$R(\theta) = \frac{\cos \theta - (A - B \sin^2 \theta)}{\cos \theta + (A - B \sin^2 \theta)} \tag{4}$$

where θ is the angle of incidence of a plane wave (cf. Fig. 4)

$$u_{inc} = e^{ik(x \sin \theta - y \cos \theta)} \quad ,$$

incident upon the boundary at $y = 0$.

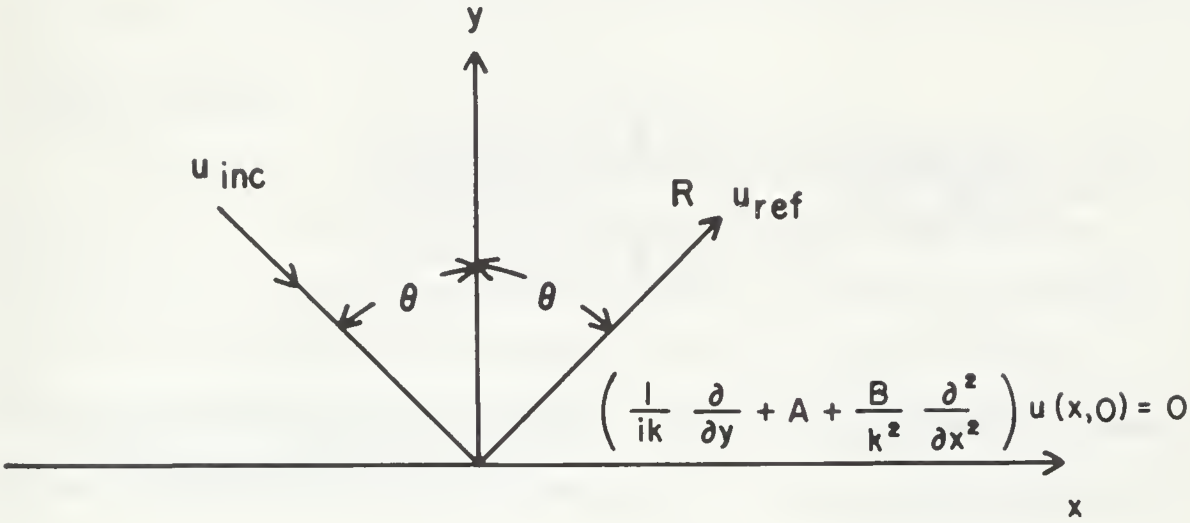

y
u inc
R u ref
θ
θ
$\left(\frac{1}{ik} \frac{\partial}{\partial y} + A + \frac{B}{k^2} \frac{\partial^2}{\partial x^2} \right) u(x,0) = 0$
x

Figure 4: Plane wave incident and reflected at $y = 0$.

To derive the expression (4) one assumes a geometrically reflected field

$$u_{ref} = Re^{ik(x \sin \theta + y \cos \theta)}$$

and the amplitude coefficient $R(\theta)$ is determined so as to make the total field

$$u_{inc} + R\, u_{ref}$$

satisfy the boundary condition (2).

An approximation procedure is then evident: Find a boundary condition whose reflection coefficient is a suitable approximation to the appropriate Fresnel reflection coefficient. One is reminded that if u represents E_Z the Fresnel coefficient is ([5], p. 492 ff.)

$$(5)\quad R_{TE} = \frac{\sin(\theta_o - \theta)}{\sin(\theta_o + \theta)} = \frac{\cos\theta - \frac{k_2}{k_1}\sqrt{1 - \left(\frac{k_1}{k_2}\right)^2 \sin^2\theta}}{\cos\theta + \frac{k_2}{k_1}\sqrt{1 - \left(\frac{k_1}{k_2}\right)^2 \sin^2\theta}}$$

where θ_o is the angle of refraction and k_2 is the propagation constant in the lower media $y \leq 0$. If $u = H_z$ then the reflection coefficient is

$$(6)\quad R_{TM} = \frac{\tan(\theta - \theta_o)}{\tan(\theta + \theta_o)} = \frac{\cos\theta - \frac{k_1}{k_2}\sqrt{1 - \left(\frac{k_1}{k_2}\right)^2 \sin^2\theta}}{\cos\theta + \frac{k_1}{k_2}\sqrt{1 - \left(\frac{k_1}{k_2}\right)^2 \sin^2\theta}}$$

In (5) and (6) we assume the magnetic permeabilities μ_1 and μ_2 to be identical for convenience only. If we compare (5) and (6) with (4) then we see that we can approximate (5) and (6) at least for all real angles of incidence by choosing A and B properly. To make this selection we note that if

$$(7)\quad \begin{cases} A_{TE} = \dfrac{k_2}{k_1} = n \\[2ex] A_{TM} = \dfrac{k_1}{k_2} = \dfrac{1}{n} \end{cases}$$

then the approximation (4) will agree exactly with either (5) or (6) for normal incidence (i.e., $\theta = 0$). For glancing incidence ($\theta = \pi/2$), the expression (4) will agree with (5) or (6) regardless of the choice of B since both are equal to -1 for $\theta = \pi/2$.

A. Angular Matching

We can choose B in many ways to improve the approximation. For example B may be chosen so as to make the approximation exact for some angle θ_1 where $0 < \theta_1 < \pi/2$. There is nothing to suggest that any angle θ_1 is to be preferred; for the balance of Section 1 we shall discuss possible choices and their motivations. For any choice of θ_1 the $B_{TE,TM}$ are determined by matching

$$1 - \frac{B_{TE,TM}}{A_{TE,TM}} \sin^2\theta_1 \quad \text{with} \qquad \sqrt{(1-(1/n)^2\sin^2\theta_1}$$

where $n = k_2/k_1$ is the index of refraction or

$$B_{TE,TM}(\theta_1) = \frac{A_{TE,TM}\left[1 - \sqrt{(1-(1/n)^2\sin^2\theta_1)}\right]}{\sin^2\theta_1} \tag{8}$$

B. Binomial Matching

If $n > 1$ we can expand the radical in (8) by the binomial theorem to obtain from (7)

$$\left\{\begin{aligned} B_{TE} &= n\left[\frac{1}{2n^2} - \frac{1}{8}\left(\frac{1}{n}\right)^4 \sin^2\theta_1 + \ldots\right] \\ B_{TM} &= \frac{1}{n}\left[\frac{1}{2n^2} - \frac{1}{8}\left(\frac{1}{n}\right)^4 \sin^2\theta_1 + \ldots\right] \end{aligned}\right. \tag{9}$$

.

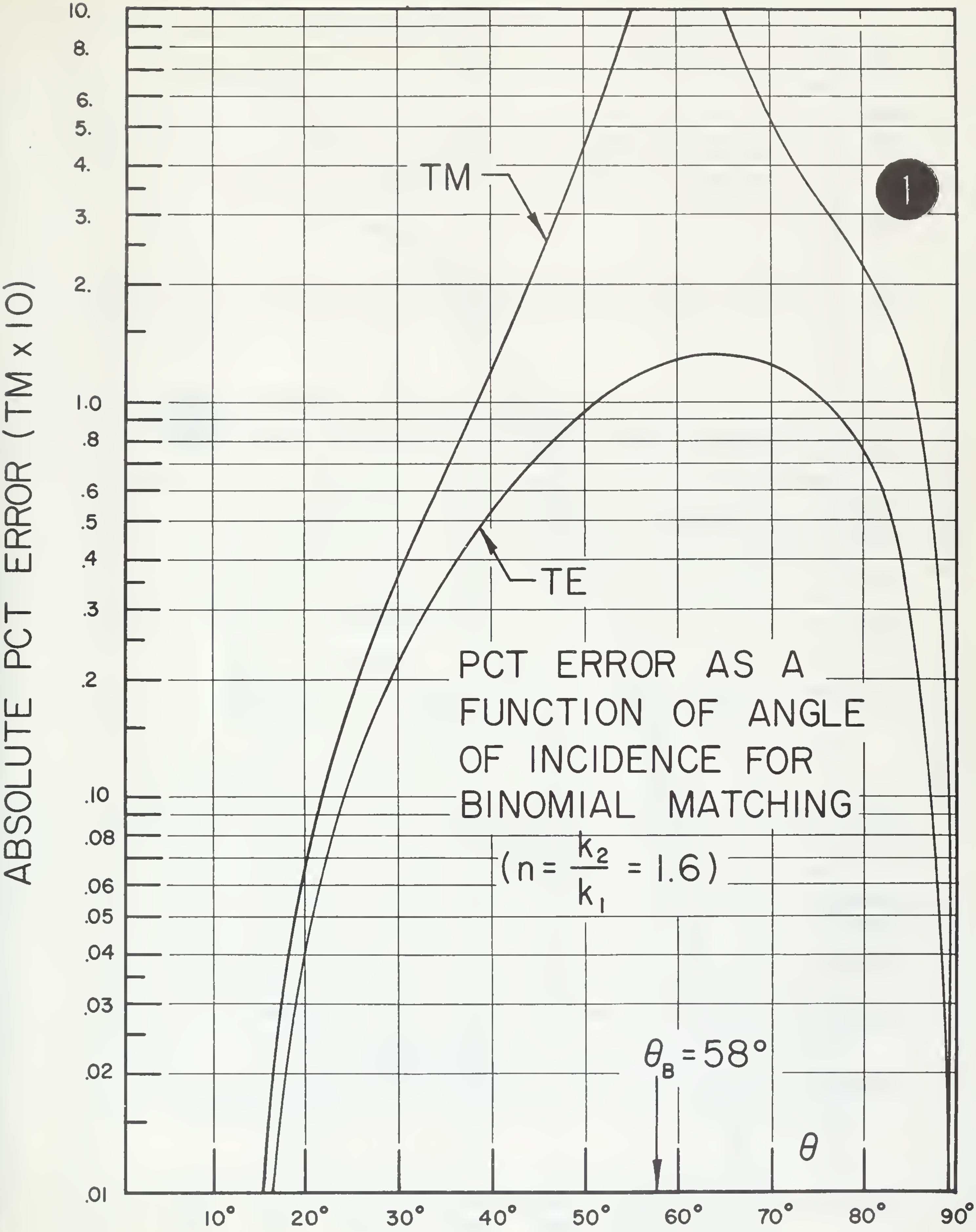
Note from (9) that the leading terms of B_{TE} and B_{TM} are independent of θ_1, the fitting angle. For analytical purposes it is very convenient to choose the B's as just these leading terms

$$(10) \qquad \begin{aligned} B_{TE} &= \frac{1}{2n} \\ B_{TM} &= \frac{1}{2n^3} \end{aligned}$$

The selection (10) will be referred to as binomial matching. Physically it corresponds to improving the match of the reflection coefficients in the neighborhood of $\theta = 0$, or normal incidence. This is illustrated in Graph 1 where we plot the percentage error between the reflection coefficients (5) and (6) and their approximations (4) with binomial-match coefficients for $n = 1.6$. Note that in the neighborhood of Brewster's angle θ_B, $R(\theta_B) = 0$ and, hence, the percentage error (but not the absolute) for TM approximation is necessarily unbounded. In Graph 2 we plot the absolute errors and we see that the absolute errors in the vicinity of θ_B remain small. Indeed the absolute errors for both the TE and TM approximations have about the same values.

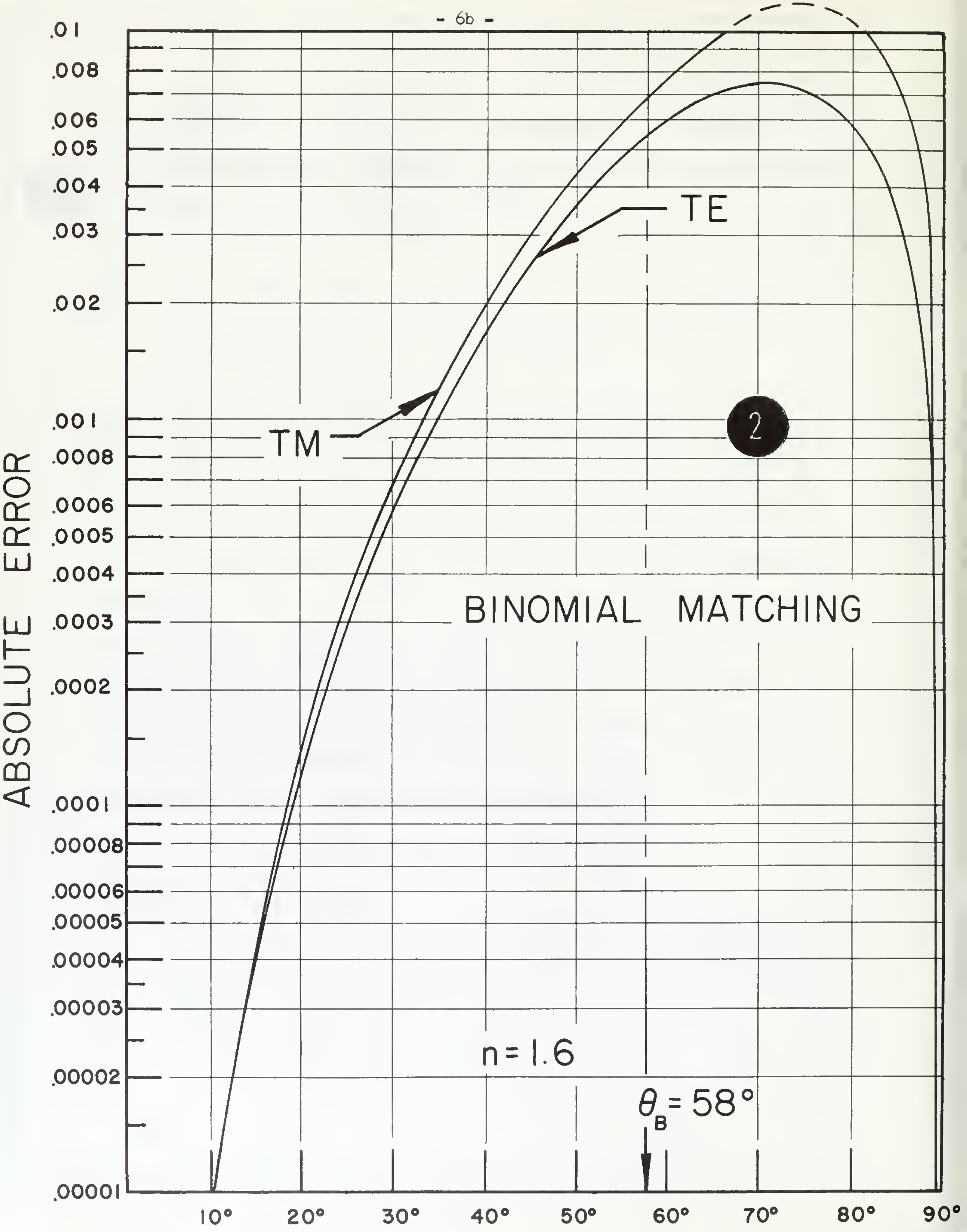
C. Brewster's Angle Matching

We can eliminate the unbounded percentage errors for the TM case in the vicinity of Brewster's angle by choosing B so as to yield a perfect match at Brewster's angle $\theta_B = \tan^{-1} n$ in which case (8) becomes



PCT ERROR AS A FUNCTION OF ANGLE OF INCIDENCE FOR BINOMIAL MATCHING

$(n = \frac{k_2}{k_1} = 1.6)$

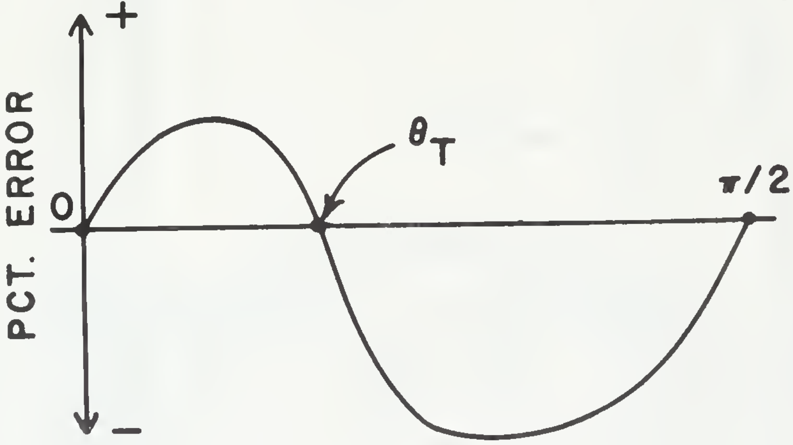

ABSOLUTE ERROR
.01
.008
.006
.005
.004
.003
.002
.001
.0008
.0006
.0005
.0004
.0003
.0002
.0001
.00008
.00006
.00005
.00004
.00003
.00002
.00001
10°
20°
30°
40°
50°
60°
70°
80°
90°
TE
TM
2
BINOMIAL MATCHING
n = 1.6
$\theta_B = 58°$

$$(11) \qquad B_{TE,TM}(\theta_B) = A_{TE,TM}\,\frac{n^2+1}{n^2}\left[1-\sqrt{1-\frac{1}{n^2+1}}\right] \approx \frac{A_{TE,TM}}{2(n^2+1)} \quad .$$

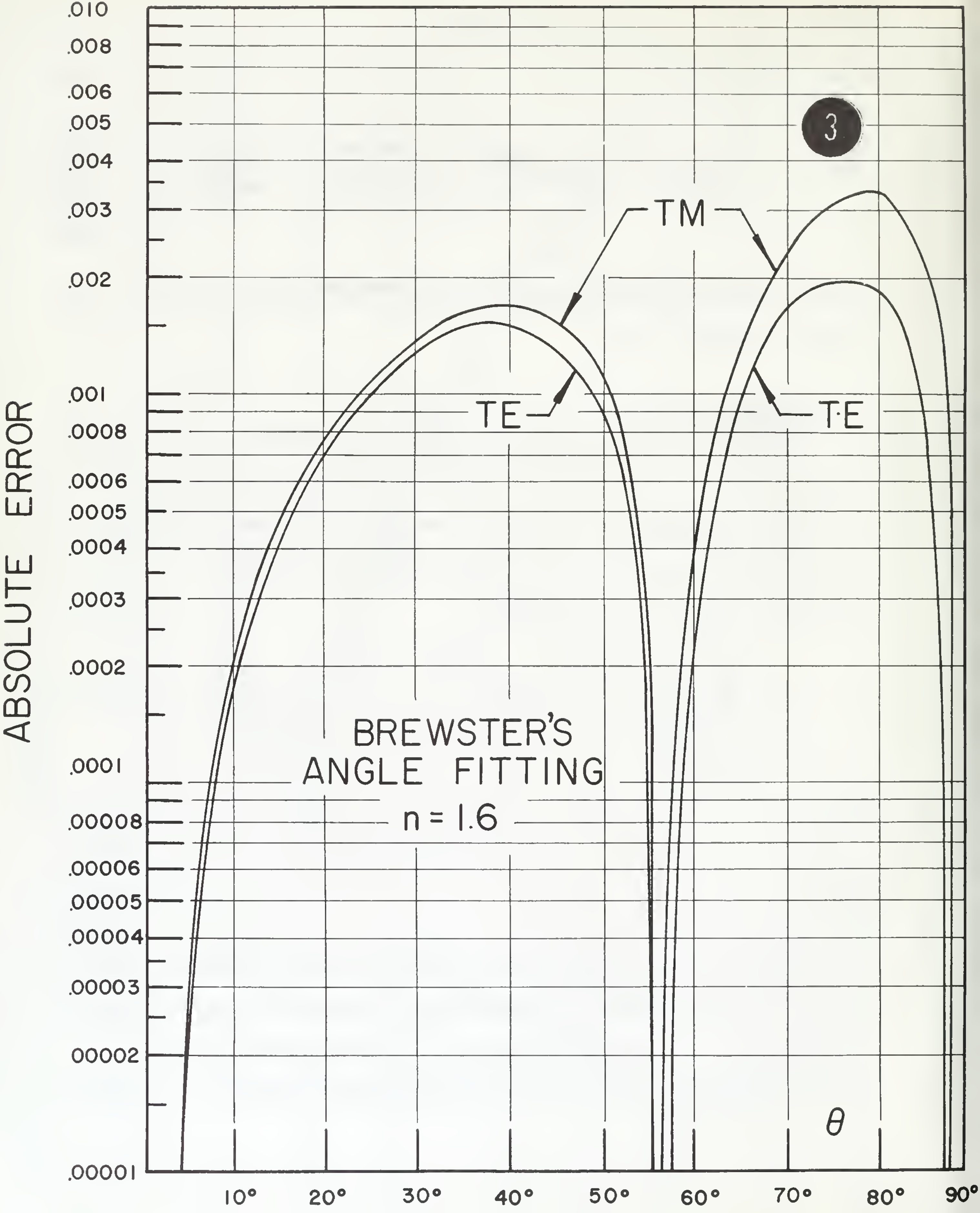
Curves 3 and 4 illustrate the percentage and absolute errors for this choice of θ_1 for $n = 1.6$.

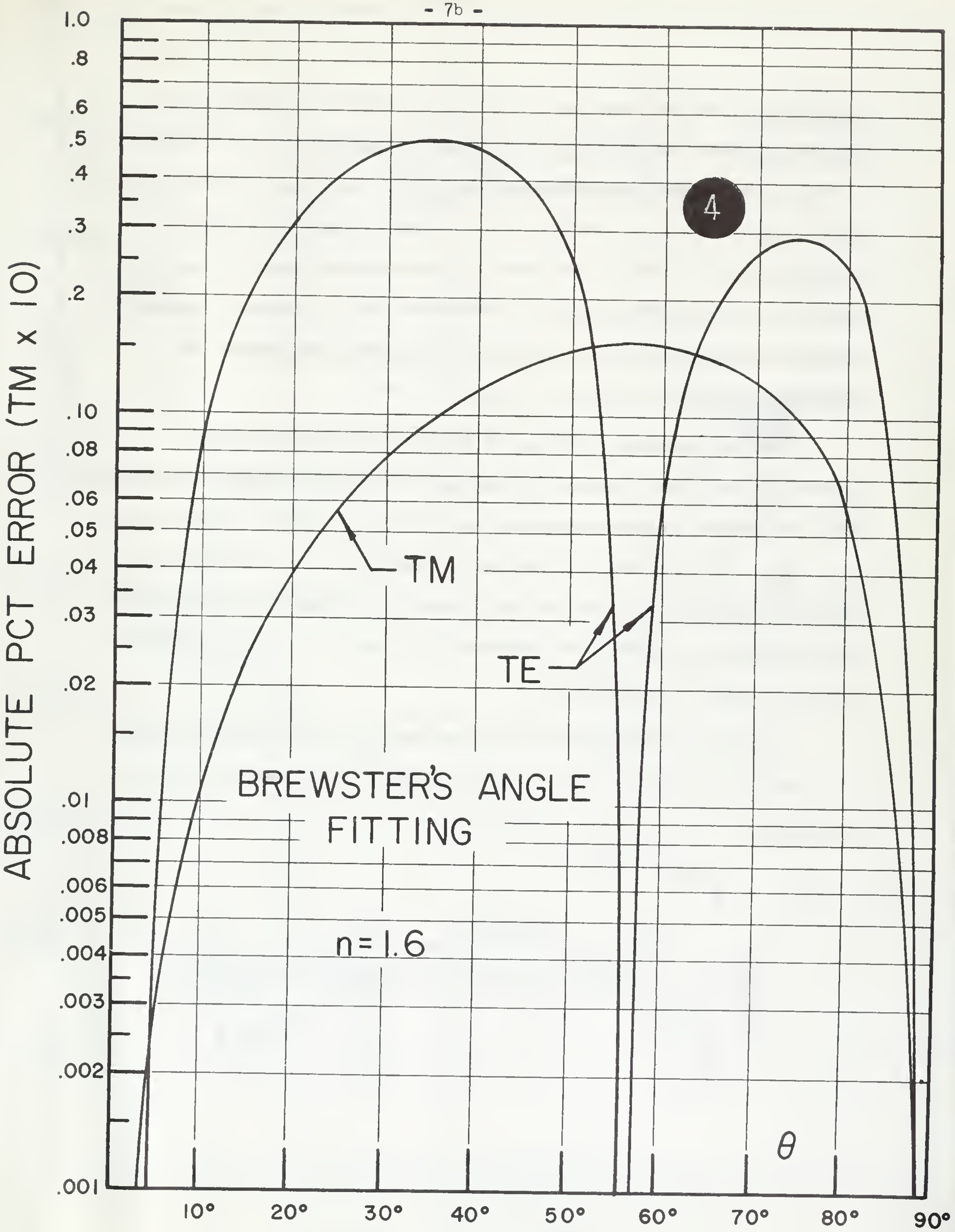
D. Chebyshev Matchings

Still another choice for B is possible at least for the TE approximation; this selection is a value of B so chosen as to minimize the maximum percentage error for the TE reflection coefficient. We note that the magnitude of the error changes sign for some value θ_T;



if we choose a value of θ_T such that the maximum positive percentage error equals the maximum negative percentage error then we shall have the least maximum error for the interval $0 \leq \theta \leq \pi/2$ by the Chebyshev criterion. The mathematical analysis to find the optimum Chebyshev matching would be

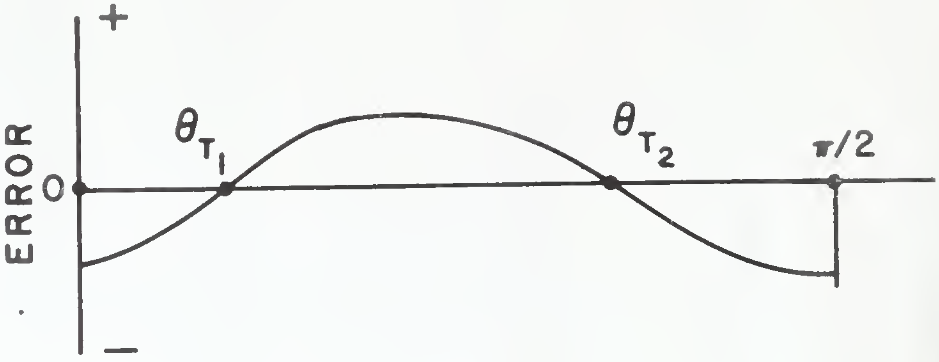

③
ABSOLUTE ERROR
.010
.008
.006
.005
.004
.003
.002
.001
.0008
.0006
.0005
.0004
.0003
.0002
.0001
.00008
.00006
.00005
.00004
.00003
.00002
.00001
TM
TE
T·E
BREWSTER'S ANGLE FITTING n = 1.6
θ
10°
20°
30°
40°
50°
60°
70°
80°
90°

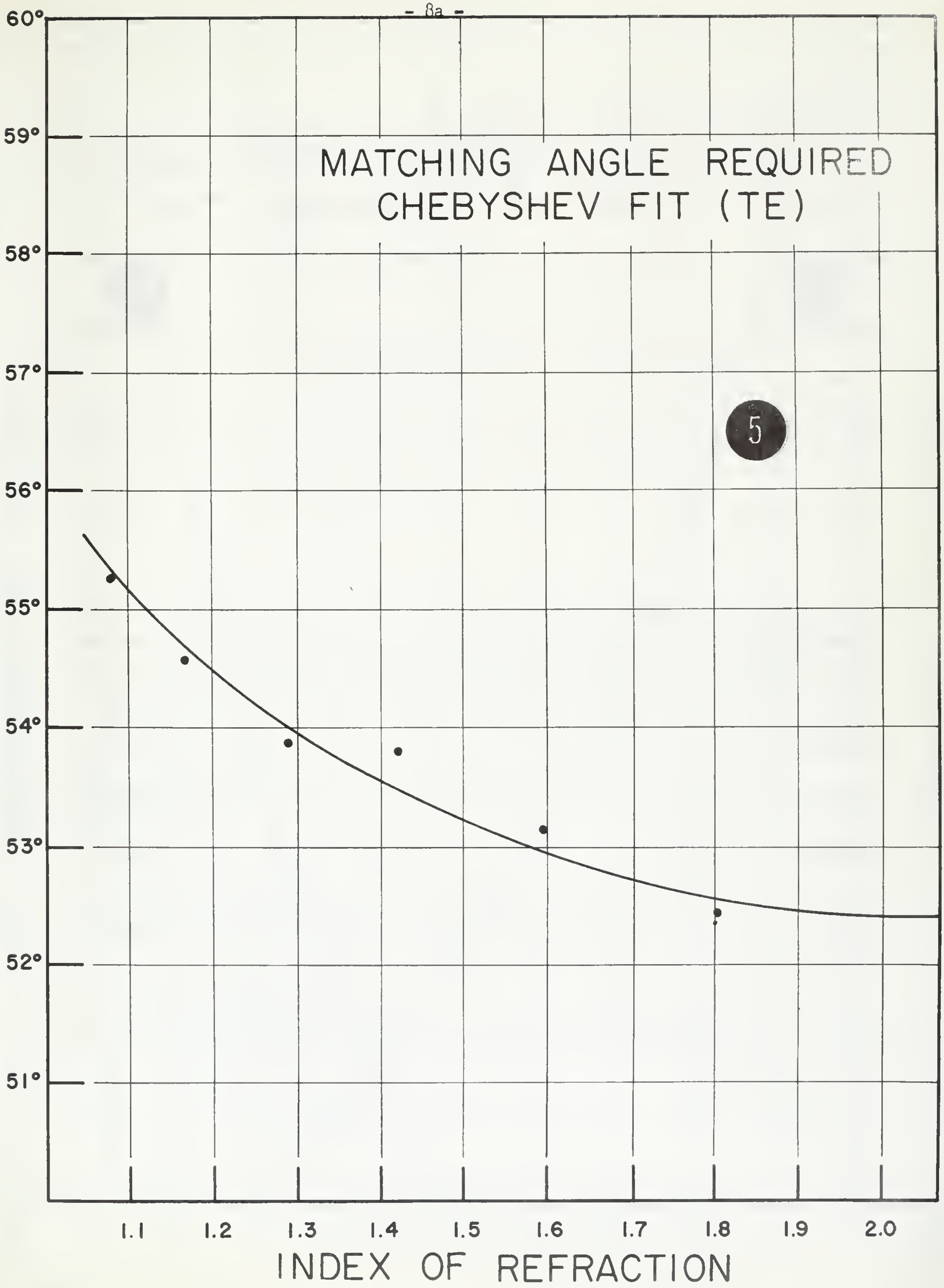

ABSOLUTE PCT ERROR (TM x 10)
1.0
.8
.6
.5
.4
.3
.2
.10
.08
.06
.05
.04
.03
.02
.01
.008
.006
.005
.004
.003
.002
.001
10°
20°
30°
40°
50°
60°
70°
80°
90°
θ
TM
TE
BREWSTER'S ANGLE FITTING
n = 1.6
4

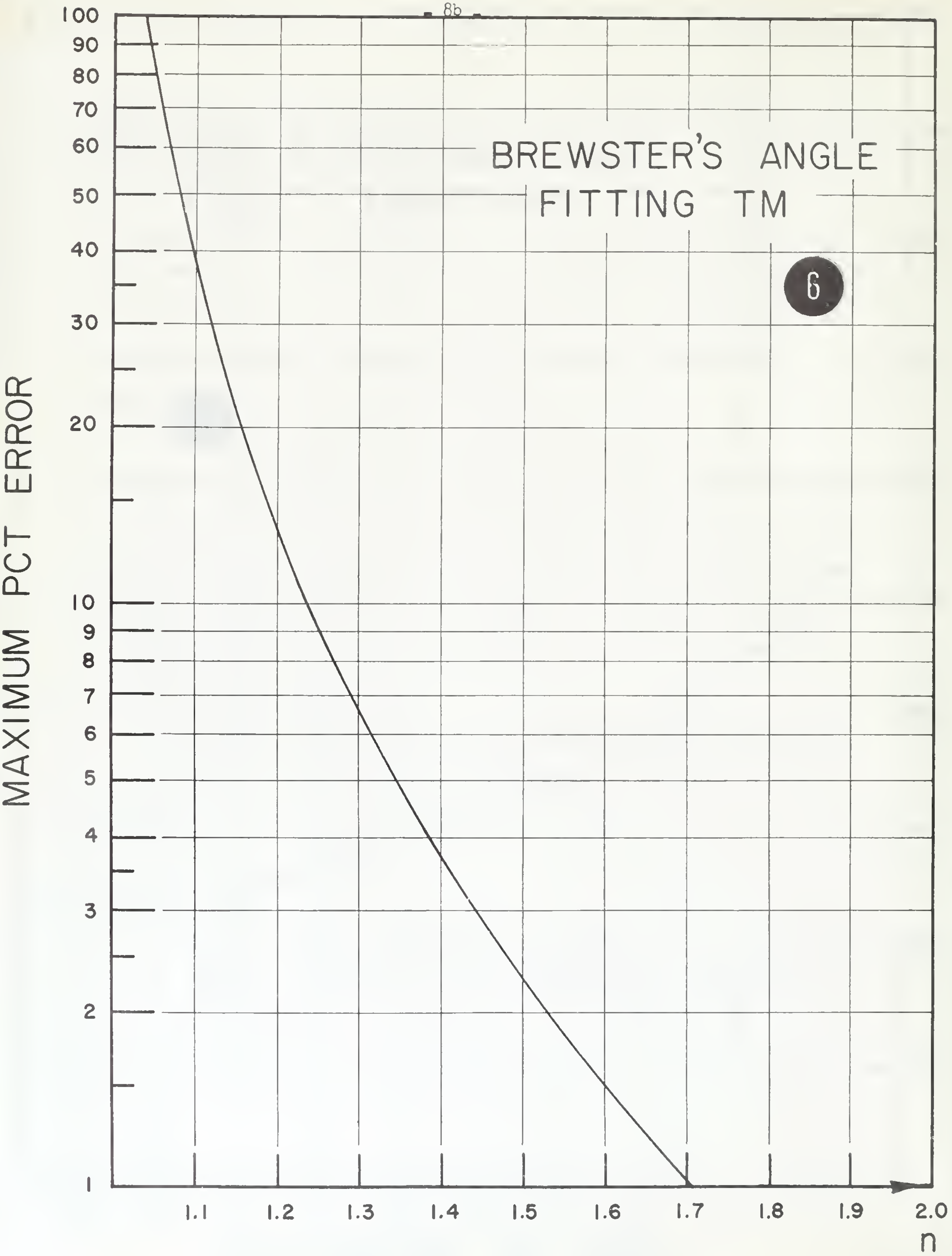
interesting. Here we have done it by a trial and error process; the values of θ_T required to afford this match for the TE case are plotted in Graph 5. The maximum associated errors for the reflection coefficient are plotted in Graph 7 as a function of n. It is not possible to repeat this procedure for the TM case owing to the unbounded nature of the percentage error in the vicinity of Brewster's angle, however a Chebyshev matching is possible for both the TE and TM cases by minimizing the *absolute* error, but we have not carried this out.

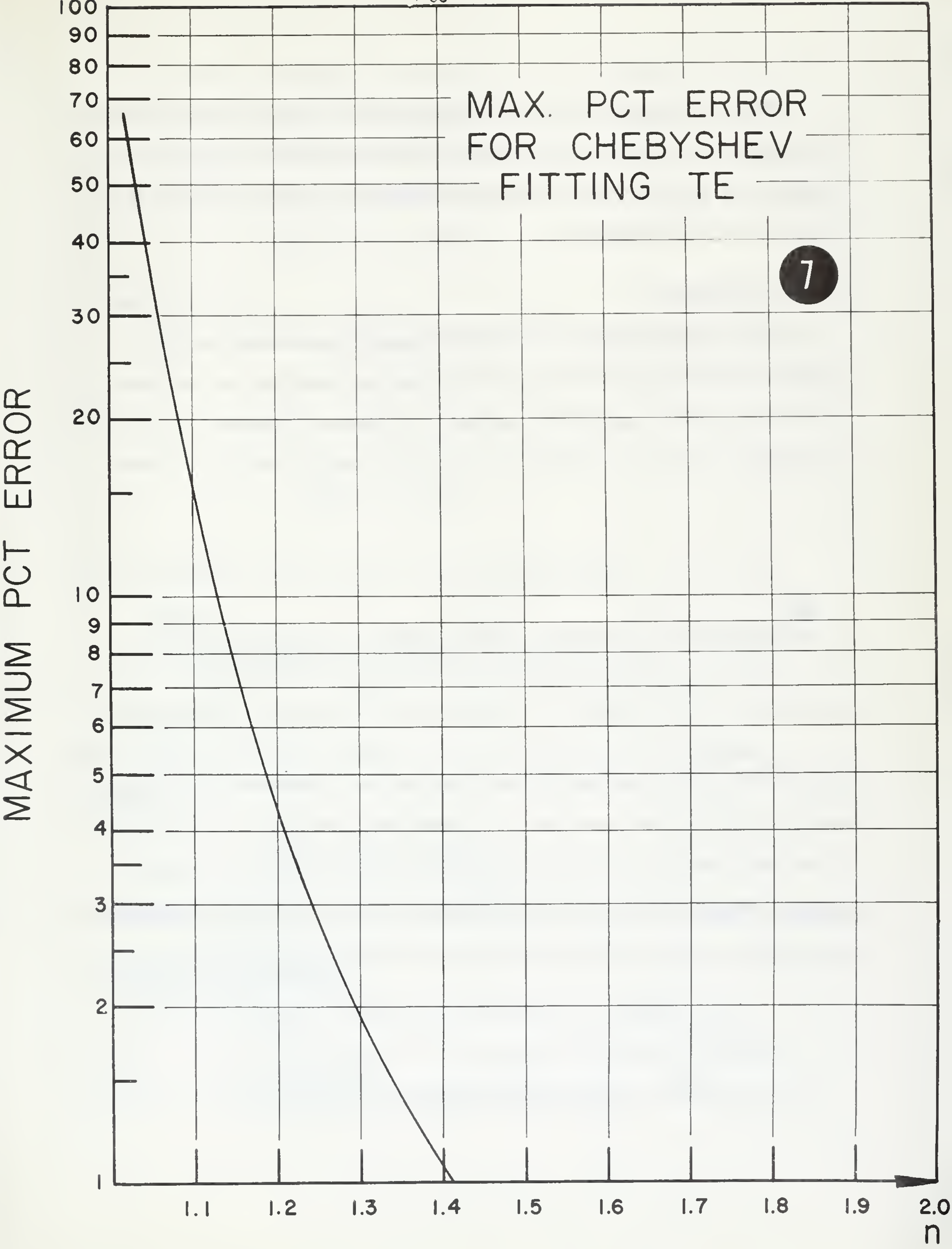
It seems reasonable to use the boundary condition (4) with Brewster's angle coefficients for problems of transverse magnetic excitation. For a problem of transverse electric excitation we can use the values of θ_T in curve (5) in formula (8) to yield a Chebyshev fit. The maximum percentage errors that arise for real angles of incidence for these two approximations are plotted for reference in curves 6 and 7 as a function of n, the index of refraction.

Even better Chebyshev fittings are possible by relaxing the demand of a perfect fit at $\theta = 0$; this permits choosing both A and B to improve the match.




MATCHING ANGLE REQUIRED
CHEBYSHEV FIT (TE)
⑤
60°
59°
58°
57°
56°
55°
54°
53°
52°
51°
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
INDEX OF REFRACTION


BREWSTER'S ANGLE
FITTING TM
6
MAXIMUM PCT ERROR
100
90
80
70
60
50
40
30
20
10
9
8
7
6
5
4
3
2
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
n


MAX. PCT ERROR
FOR CHEBYSHEV
FITTING TE
7
MAXIMUM PCT ERROR
100
90
80
70
60
50
40
30
20
10
9
8
7
6
5
4
3
2
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
n

This procedure would yield a perfect fit for four values of θ over the interval $0 \leq \theta \leq \pi$. This particular matching was not carried out, owing to the labor involved in doing it empirically, and the excellent fit of the cruder approximations.

E. Grazing Matching

For certain problems an alternate procedure is available if a physical inquiry is of such a nature that most of the field of interest is associated with a particular angle of incidence. An example is a transmitter and distant receiver--both located near the earth's surface.

The geometry of this problem is such that one would expect most of the energy arriving at the receiver to be associated with the arrival of rays in the neighborhood of grazing incidence. For this problem one would choose a value of B to match the derivative of the reflection coefficient at grazing incidence. From (8) we see that the requisite $B(\pi/2)$ would be

$$B_{TE,TM}(\pi/2) = A_{TE,TM}\left[1 - \sqrt{1 - \frac{1}{n^2}}\right] \tag{12}$$

In subsections A-E we have indicated several possible choices of the coefficients in the B. C. For the balance of the report we shall assume that the coefficients have been chosen for a particular problem, and confine our discussion to the question of what is true for any such choice of A and B, unless otherwise indicated.

F. Generalizations

Beside the above-mentioned methods, other refinements are possible. Thus any desired degree of accuracy can be obtained by adding additional terms to the boundary condition: By adding such terms as

$$\frac{\partial^4 u}{\partial x^4}, \quad \frac{\partial^6 u}{\partial x^6}, \quad \ldots$$

to the boundary condition (4) we would have generated

$$\sin^4 \theta \quad , \qquad \sin^6 \theta, \ldots$$

in the approximate reflection coefficient. The coefficients of these terms would then be available, either to match as many terms in a series expansion of

$$\sqrt{1 - (\frac{1}{n})^2 \sin^2 \theta}$$

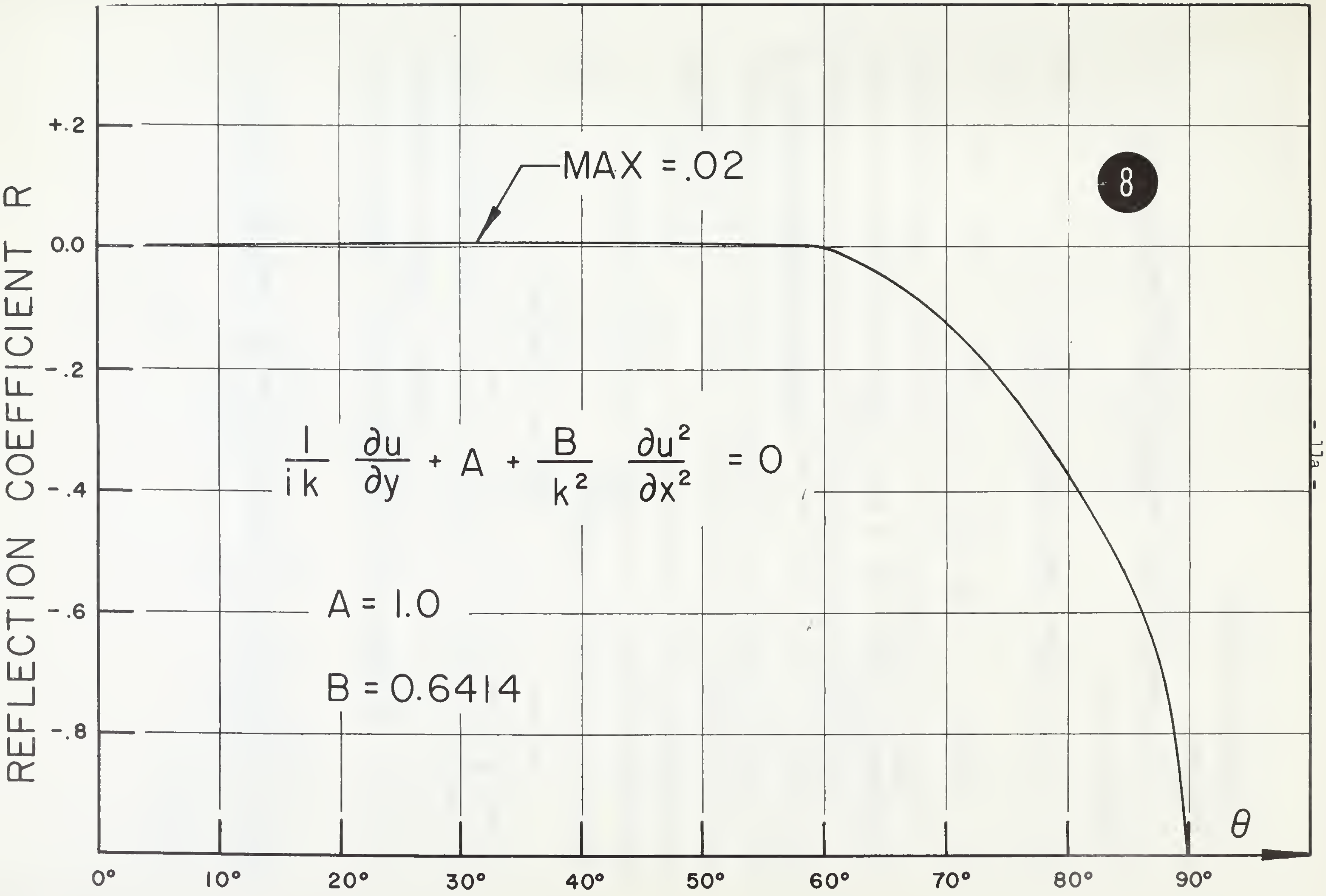
as needed for superlative accuracy, or for use in a Chebeyshev approximation.

2. Limitations and Remarks

Our approximation procedure is limited by the restriction $k_2 > k_1$, or $n > 1$. For if $k_2 < k_1$ then, there arises a real angle of incidence corresponding to total reflection for which the second radical in (1.5) and (1.6) becomes imaginary. Since (1.4) involves constant coefficients the boundary condition is unable to reproduce the transition from a real reflection coefficient to a complex one for real angles of incidence. This is clear by inspection of the reflection coefficient (1.4). So long as both A and B are real, the reflection coefficient remains real for any real angle θ. If A and B are imaginary or complex, then $R(\theta)$ will always be complex for any range of θ.

One can also observe from inspection of the curves 6 and 7 that the approximation deteriorates for $n \backsim 1$. Physically, this corresponds to an interface with a reflection coefficient in the vicinity of zero and hence difficult to approximate on a percentage basis. However, this fact allows one to use other approximate methods[9]. However, the absolute errors remain small and bounded. As a consequence the boundary condition (1.4) specialized for $n = 1$ may be useful as a model of a perfect absorber since the reflection coefficient it defines is so small for most of the range. A curve of this case is drawn in Graph 8. It is unavoidable that the reflection coefficient goes to -1 for $\theta = \pi/2$. This behavior is forced by the form of (1.4) which is true even in the exact case if $n \neq 1$.

A special feature of the new boundary condition is that it involves a second derivative in x. In Section 4 we show that introduction of such a term does not violate reciprocity, and the method of demonstration will show that we can include additional derivatives with respect to x of even order without sacrificing reciprocity.


8
MAX = .02
$\frac{1}{ik}\frac{\partial u}{\partial y} + A + \frac{B}{k^2}\frac{\partial u^2}{\partial x^2} = 0$
A = 1.0
B = 0.6414
REFLECTION COEFFICIENT R
+.2
0.0
-.2
-.4
-.6
-.8
0°
10°
20°
30°
40°
50°
60°
70°
80°
90°
θ

3. Existence and Uniqueness

A. Existence

In this section we shall obtain an explicit solution to the following problem

$$(1)\quad (\nabla^2 + k^2)\, u(x,y) = 4i\delta(x)\,\delta(y-h), \qquad y \geq 0$$

$$(2)\quad \left(\frac{1}{ik}\frac{\partial}{\partial y} + A + \frac{B}{k^2}\frac{\partial^2}{\partial x^2}\right) u = 0, \qquad y = 0$$

$$(3)\quad \lim_{\rho \to \infty} \sqrt{\rho}\left(\frac{\partial u}{\partial \rho} - iku\right) = 0\,, \qquad y \geq 0\,,\quad 0 \leq \theta \leq \pi\quad .$$

$$\rho = \sqrt{(x^2 + y^2)}\,, \qquad \theta = \arctan\,(y/x)$$

Once a solution to this problem has been obtained and discussed we shall show that we have the only such solution. The purpose of this demonstration and that of Section 4 in which reciprocity is proved[*] is to show that the introduction of the new boundary condition (2) leads to a well-defined approximate formulation of the standard problem of the radiating line source above a dielectric interface.

A solution to the above problem can be obtained very simply by expressing the excitation field $u_e(x,y)$ by separation of variables, in the form

$$(4)\quad u_e(x,y) = H_0^{(1)}(k\rho_h) = -\frac{1}{\pi}\int_C \frac{e^{i\nu x + i\sqrt{k^2-\nu^2}\,|y-h|}}{\sqrt{k^2-\nu^2}}\,d\nu$$

where $\rho_h = \sqrt{x^2 + (y-h)^2}$ and expressing the secondary image field $u_s(x,y)$ in a similar form, namely,

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[*] page 22.

$$(5)\quad u_s(x,y) = -\frac{1}{\pi}\int_C \frac{G(\nu)\, e^{i\nu x + i\sqrt{k^2-\nu^2}\,|y+h|}}{\sqrt{k^2-\nu^2}}\, d\nu \quad .$$

Here $G(\nu)$ is to be determined by the condition that the total field (6)

$$(6)\quad u_T(x,y) = u_e(x,y) + u_s(x,y)$$

satisfies the boundary condition at $y = 0$. The contour C is taken in the complex ν-plane along the real axis as shown in Fig. 8 which also indicates the choice of branch cuts for $\sqrt{k^2 - \nu^2}$. In this section we assume k to be real unless otherwise mentioned.

If we apply the boundary condition (2) to the combination (6) we find that if we choose

$$(7)\qquad G(\nu) = \frac{\dfrac{\sqrt{k^2-\nu^2}}{k} - \left(A - \dfrac{B\nu^2}{k^2}\right)}{\dfrac{\sqrt{k^2-\nu^2}}{k} + \left(A - \dfrac{B\nu^2}{k^2}\right)}$$

then the boundary condition (2) will be satisfied. It will follow from the asymptotic development of the integrals (5) that the radiation condition (3) will be satisfied. The complete solution can be displayed as

$$(8)\ u_T(x,y) = H_0^{(1)}(k\rho_h) - \frac{1}{\pi}\int_C \frac{\dfrac{\sqrt{k^2-\nu^2}}{k} - \left(A - \dfrac{B\nu^2}{k^2}\right)}{\dfrac{\sqrt{k^2-\nu^2}}{k} + \left(A - \dfrac{B\nu^2}{k^2}\right)} \cdot \frac{e^{i\nu x + i\sqrt{k^2-\nu^2}\,|y+h|}}{\sqrt{k^2-\nu^2}}\, d\nu \ .$$

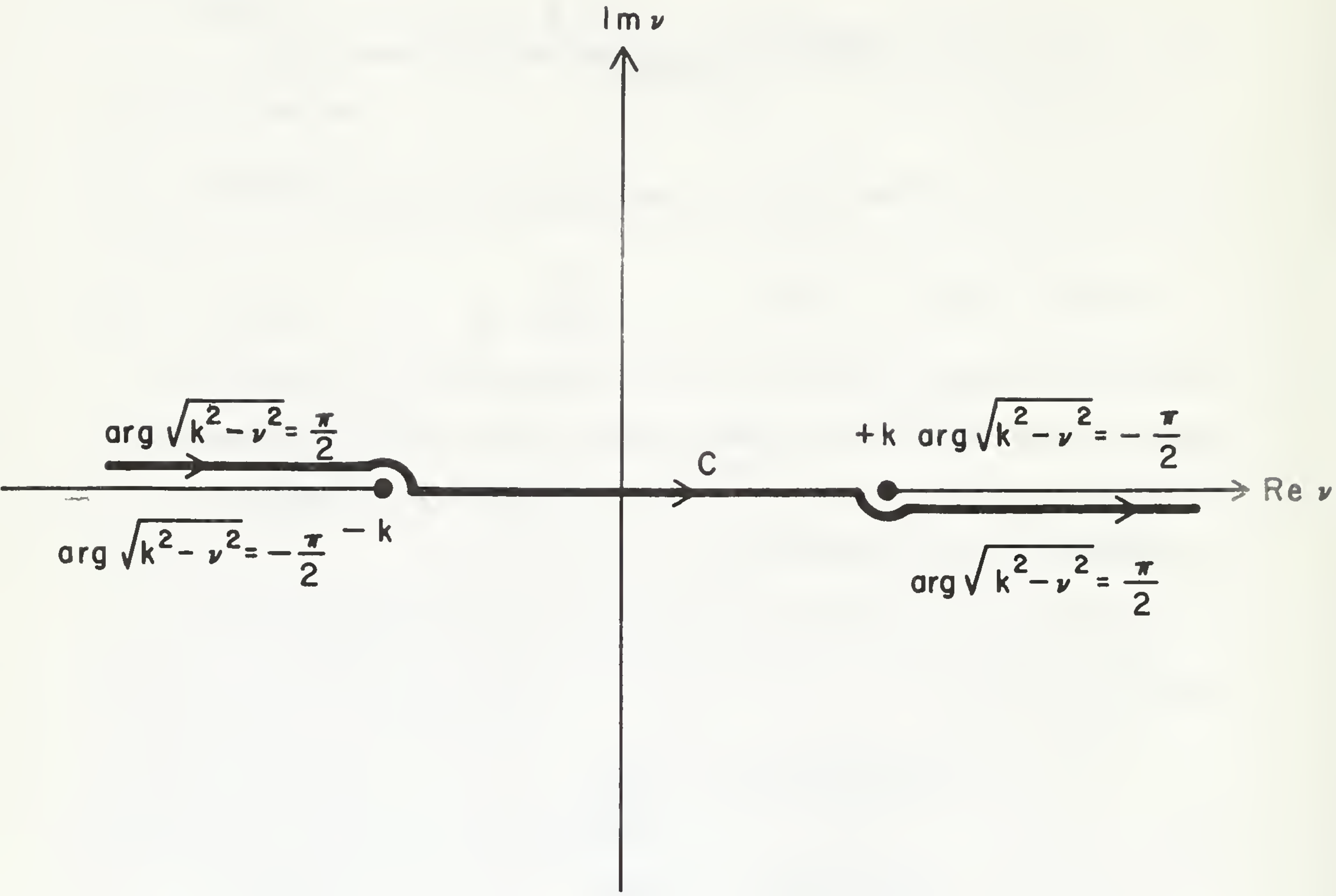

Figure 5: The contour C in the ν-plane defining (8). In this illustration the imaginary part of k is neglected.

In order to discuss the integral (5) that appears in (8) it is convenient to introduce the polar coordinates

$$(9a)\qquad \begin{cases} \rho_{-h} = \sqrt{x^2 + (y + h)^2} \\ \theta = \tan^{-1} \dfrac{y + h}{x} \end{cases},$$

or

$$(9b)\qquad \begin{cases} x = \rho_{-h} \cos \theta \\ y + h = \rho_{-h} \sin \theta \end{cases}.$$

It is also convenient to make a transformation from the ν-plane to the $\cos \phi$ plane; where

$$(10)\qquad \nu = k \cos \phi \,.$$

When this is done, then, in the region $x \geq 0$, $y \geq 0$, the integral (5) becomes

$$(11)\qquad u_s(\rho_{-h}, \theta) = \frac{1}{\pi} \int_{C'(\phi)} R(\phi)\, e^{ik\rho_{-h}\cos(\phi - \theta)}\, d\phi$$

where $R(\phi)$ is the approximate reflection coefficient (1.4) extended to the complex ϕ- plane

$$(12)\qquad R(\phi) = \frac{\sin \phi - (A - B \cos^2 \phi)}{\sin \phi + (A - B \cos^2 \phi)} \,.$$

Owing to the restricted choices of A and B as discussed in this report (Section 1), there are no roots of the denominator* of $R(\emptyset)$ for real $\emptyset$ in the range $0 \leq \emptyset \leq \pi/2$. Consequently we can apply the method of saddle points (see Part II for a complete discussion) directly to (11) to yield the asymptotic development

$$(13) \qquad u_s(\rho_{-h}, \theta) = \sqrt{\frac{2}{\pi k \rho_{-h}}}\; R(\theta)\, e^{ik(\rho_{-h} - \frac{\pi}{4})} + O\left[(k\rho_{-h})^{-\frac{3}{2}} \right] .$$

From the development of the Hankel function

$$(14) \qquad H_o^{(1)}(k\rho_h) = \sqrt{\frac{2}{\pi k \rho_h}}\; e^{ik(\rho_h - \frac{\pi}{4})} + O\left[(k\rho_h)^{-\frac{3}{2}} \right]$$

it follows that the entire field at large distances is

$$(15) \quad u(\rho,\theta) = \sqrt{\frac{2}{\pi k \rho}} \left[1 + R(\theta)\, e^{ik(\rho_{-h} - \rho_h)} \right] e^{ik(\rho_h - \frac{\pi}{4})}$$

plus terms of order $\rho_h^{-\frac{3}{2}}$. For $\rho_{-h} \gg 1$ we have that

$$(16) \qquad \rho_{-h} - \rho_h = h \sin \theta$$

so that (15) becomes

$$(17) \qquad u(\rho,\theta) = \sqrt{\frac{2}{\pi k \rho_h}} \left[1 + R(\theta)\, e^{ikh \sin \theta} \right] e^{ik(\rho_h - \pi/4)}$$

which we see is just (4.5) within a constant factor if we note that the angle used in this section is the complement of the angle used in Section 4.

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*see Appendix C

Hence the field so obtained possesses the predicted properties of equation (3) of this section. Equations (1), (2) have been fulfilled automatically in the course of the derivation.

B. Uniqueness

We can utilize the far-field behavior (17) of u(x,y) to prove the uniqueness theorem: Let u(x,y) be any function which satisfies

$$(18) \qquad (\nabla^2 + k^2)\, u(x,y) = \delta(x-x_o)\, \delta(y-y_o) \, , \qquad y \geq 0$$

and

$$(19) \qquad \left(\frac{1}{i} \frac{\partial}{\partial y} + A + B \frac{\partial^2}{\partial x^2} \right) u(x,o) = 0 \, , \qquad y = 0$$

where A and B are arbitrary real constants.

In addition we shall stipulate that u(x,y) can be developed as

$$(20) \; u(\rho,\theta) = \begin{cases} g(\theta) \dfrac{e^{ik\rho}}{\sqrt{k\rho}} + O\left[(k\rho)^{-\frac{3}{2}}\right], & \theta \varepsilon M \\ O(1) & \theta \varepsilon \overline{M} \end{cases}$$

for $k\rho \gg 1$ where M is some set whose complement $\overline{M}$ is a set of measure zero on the interval $0 \leq \theta \leq \pi$.

Proof

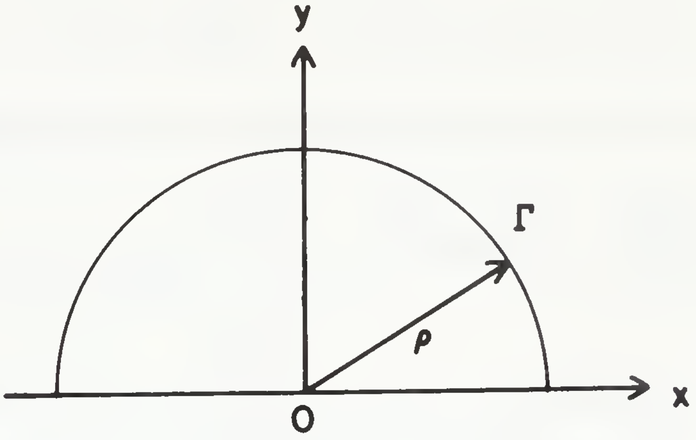
Let $u(x,y) = u_1(x,y) - u_2(x,y)$ where u_1 and u_2 are any two functions that satisfy (18), (19) and (20). Then u(x,y) is a solution of the homogeneous wave equation

$$(21) \qquad (\nabla^2 + k^2)\, u(x,y) = 0$$

and in addition satisfies (19) and (20). Construct the function $v(x,y)$ in terms of $u(x,y)$ as

$$(22) \quad v(x,y) \equiv \frac{1}{i}\frac{\partial u}{\partial y} + Au + B\frac{\partial^2 u}{\partial x^2}, \qquad y \geq 0$$

then $v(x,y)$ satisfies Dirichlet conditions at $y = 0$ as well as the homogeneous wave equation for $y > 0$; we shall first show that v must vanish identically for $y \geq 0$. This can be seen very simply by an application of Green's theorem about a semi-circle Γ of radius ρ to $v(x,y)$ and its conjugate $v^*(x,y)$.



Namely we have

$$\int_{\Gamma} \left(v^* \frac{\partial v}{\partial n} - v\frac{\partial v^*}{\partial n}\right) ds = \int_{A(\Gamma)} \left(v^*\nabla^2 v - v\nabla^2 v^*\right) d\tau ,$$

and by virtue of the boundary condition (19) (which implies that $v^*(x,o) \equiv v(x,o) \equiv 0$) it follows that

$$(23) \quad \int_0^{\pi} \left(v^* \frac{\partial v}{\partial \rho} - v \frac{\partial v^*}{\partial \rho}\right) \rho d\theta = \int_{A(\Gamma)} (v^* \nabla^2 v - v \nabla^2 v^*) \, d\tau \quad .$$

If k^2 has a positive imaginary part σ, the line integral will decay exponentially with increasing ρ leaving us with

$$\lim_{\rho \to \infty} \quad 2i\sigma \int_{A(\Gamma)} v^* v \, d\tau = 0$$

which implies $v \equiv 0$. If, on the other hand, k^2 is real, the integral over $A(\Gamma)$ vanishes identically which leaves us with

$$(24) \quad \int_0^{\pi} \left(v^* \frac{\partial v}{\partial \rho} - v \frac{\partial v^*}{\partial \rho}\right) \rho d\theta = 0 \quad .$$

Once again if we let $\rho \to \infty$ we can use the condition (20) to change (24) to the form

$$(25) \quad \int_0^{\pi} p(\theta) \, p^*(\theta) \, d\theta = 0$$

where $p(\theta)$ is the pattern function for $v(x,y)$; this in turn implies $p(\theta) = 0$ a.e. The function $v(x,y)$ therefore has a vanishing far-field amplitude in its region of definition $y \geq 0$. We wish to apply a theorem of Rellich in order to show $v \equiv 0$. The relevant theorem applies to a function regular outside some circle. We therefore proceed to extend v as follows: Since $v(x,y)$ vanishes on the x-axis it can be analytically continued into the lower half-plane by the reflection principle. Hence

v is now regular in the complete x, y-plane. We are now in a position to apply Rellich's Theorem. This theorem[4] states that zero is the only wave function which is regular in the entire x,y-plane, and which vanishes more rapidly than $O(\frac{1}{\sqrt{\rho}})$ for $\rho \to \infty$ and $\theta \varepsilon I$ where I is any interval. Equation (25) therefore implies that $v \equiv 0$.

These results imply that any solution $u(x,y)$ of the homogeneous wave equation with the boundary condition (19) satisfies the differential equation.

$$(26) \qquad \frac{1}{i}\frac{\partial u}{\partial y} + Au + B\frac{\partial^2 u}{\partial x^2} = 0$$

for all $y \geq 0$. It is a simple matter to show that this is inconsistent with the assumed aymptotic development (20) thus yielding the desired contradiction. To obtain this result, substitute (20) into (26) and we obtain almost everywhere

$$(27) \qquad g(\theta)\frac{e^{ik\rho}}{\sqrt{\rho}}\left[k \sin\theta + A - B\cos^2\theta\right] = O\left[\frac{e^{ik\rho}}{\rho^{3/2}}\right],$$

in which $x = \rho\cos\theta$ and $y = \rho\sin\theta$. Equation (27) can be satisfied only if $g(\theta)$ vanishes, or else, if

$$(28) \qquad k\sin\theta + A - B\cos^2\theta = 0, \qquad \text{a.e.}\ .$$

Since (28) is a quadratic in $\sin\theta$ it follows that the equation (28) can be satisfied for at most two values of $\sin\theta$. Hence

$$(29) \qquad g(\theta) = 0 \qquad \text{a.e.}, \qquad 0 \leq \theta \leq \pi.$$

The result (29) is not sufficient to imply that $u(x,y) = 0$. We shall complete the theorem by an appeal to Stokes's theorem. Since $u(x,y)$ is a wave function the integral

$$(30) \qquad \int_{A(\Gamma)} (u\nabla^2 u^* - u^*\nabla^2 u)\, d\tau = 0$$

vanishes for any region bounded by a curve Γ in the upper half plane. In particular if Γ is a semicircle of radius ρ we can compute the integral (30) in another way. We observe that if the Laplacian operates on any wave function satisfying (26) we obtain

$$\nabla^2 u = -k^2 u = \frac{1}{iA}\frac{\partial u}{\partial y} + \frac{B}{A}\frac{\partial^2 u}{\partial x^2}$$

so that (30) becomes

$$0 = \int_{A(\Gamma)} \left[-\frac{1}{iA}(u\,u_y^* + u^* u_y) + \frac{B}{A}(uu_{xx}^* - u_{xx}\,u^*) \right] d\tau$$

$$= \int_{A(\Gamma)} \left[\frac{B}{A}\frac{\partial}{\partial x}(uu_x^* - u_x u^*) - \frac{1}{iA}\frac{\partial}{\partial y}(uu^*) \right] d\tau$$

whence, by Stokes's theorem

$$0 = \int_{\Gamma} \frac{B}{A}(uu_x^* - u_x u^*)\, dy + \frac{1}{iA}\, uu^*\, dx$$

or

$$0 = I_1 + I_2 + I_3$$

where

$$I_1 = \frac{B}{A}\int_0^\pi (uu_x^* - u_x u^*)\ \rho \cos\theta\ d\theta$$

$$I_2 = -\frac{1}{iA}\int_0^\pi uu^*\ \rho \sin\theta\ d\theta$$

$$I_3 = \frac{1}{iA}\int_{-\rho}^{\rho} uu^*\ dx \quad .$$

As $\rho \to \infty$ the first two integrals I_1 and I_2 vanish since by (29) both u and u_y behave like

$$\rho^{-\frac{3}{2}}, \text{ a.e.} \quad \text{as } |\rho| \to \infty \quad 0 \leq \theta \leq \pi \ .$$

We are left with

$$\int_{-\infty}^{\infty} uu^*\ dx = \infty$$

so that u must vanish on the boundary, $y = 0$. Hence $u(x,y) \equiv 0$, $y \geq 0$ by the same argument previously applied to the function v.

4. Validity of the Approximation (with respect to the field of a line source).

In Sections 1 and 2 we have given no justification for the boundary condition (1.4) other than it guarantees that the error in the reflection coefficient is small for plane waves at real angles of incidence. The purpose of this section is to prove that if we have a line source above a dielectric

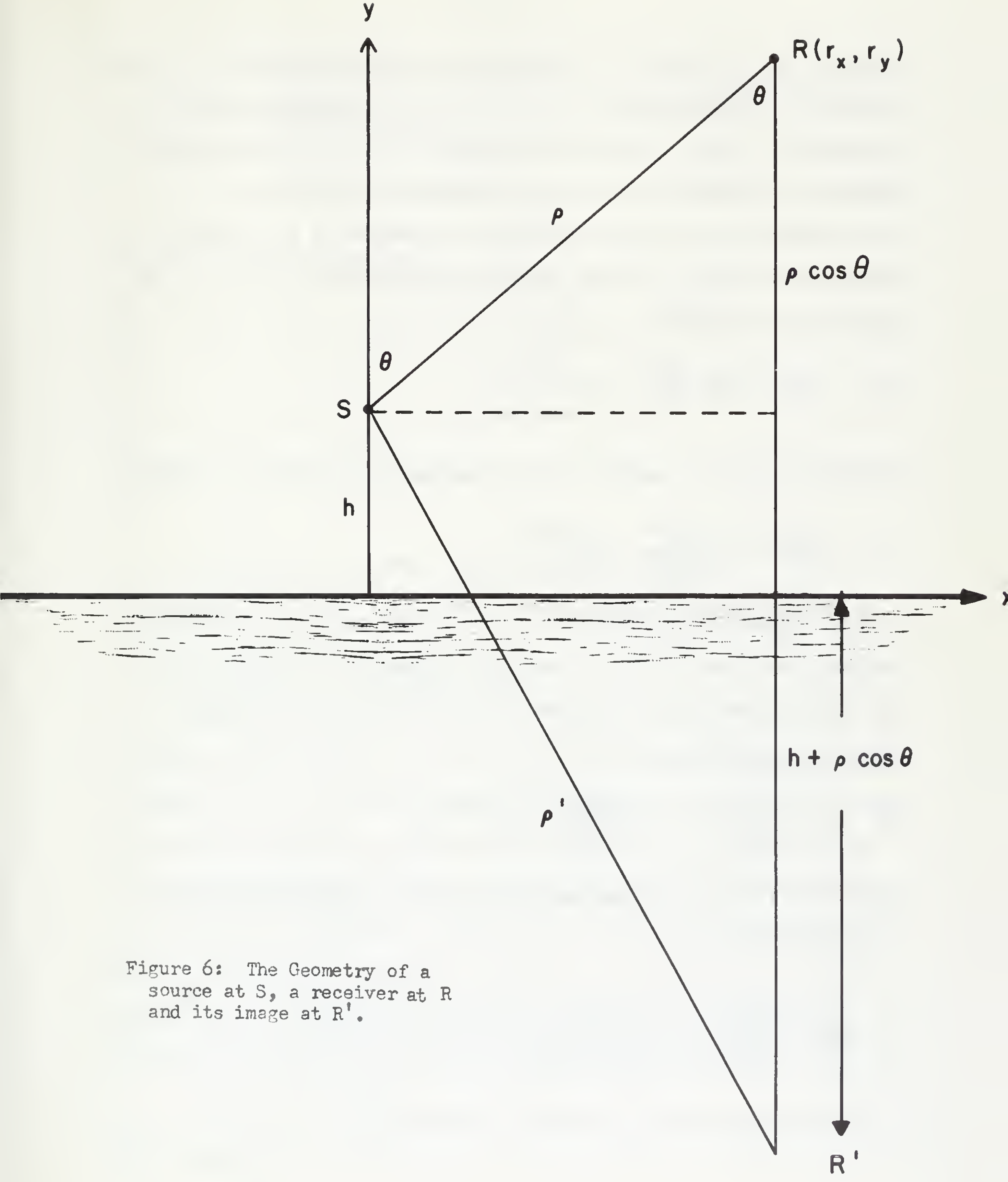


Figure 6: The Geometry of a source at S, a receiver at R and its image at R'.

interface and we replace the interface by the boundary condition (1.4) then the maximum percentage error that can arise in the far-field by this replacement is simply related to the maximum error in approximating the reflection coefficient. The proof is based upon reciprocity: If one has a source S at (0,h) above a dielectric interface at $y = 0$ and a receiver at $R = (r_x, r_y)$ in the far-field then the field at R due to the source S is of the form

$$(1) \qquad u_R(S) = g(\theta) \frac{e^{ik\rho}}{\sqrt{\rho}} + O(\rho^{-3/2})$$

and since reciprocity applies we can interchange the source and receiver

$$(2) \qquad u_S(R) = g(\theta) \frac{e^{ik\rho}}{\sqrt{\rho}} + O(\rho^{-3/2})$$

with the source in the far-field. For a distant source we know that the received field behaves like a plane wave incident at an angle θ plus $R(\theta)$ times a geometrically reflected wave where $R(\theta)$ is the Fresnel reflection coefficient, i.e.

$$(3) \quad u_S(R) = \frac{e^{ik\rho}}{\sqrt{\rho}} + R(\theta) \frac{e^{ik\rho'}}{\sqrt{\rho'}} + O(\rho^{-3/2})$$

in which ρ' is the image distance from S to R'. Hence by reciprocity $(u_S(R) = u_R(S)$ or

$$(4) \quad u_R(S) = \frac{e^{ik\rho}}{\sqrt{\rho}} + R(\theta) \frac{e^{ik\rho'}}{\sqrt{\rho'}} + O(\rho^{-3/2}) \quad .$$

For $\rho \gg 1$ we can neglect the difference $\rho' - \rho$ in the radical but must preserve it in the exponential

$$\rho' \simeq \rho + h \cos \theta , \qquad \rho \gg 1$$

so that (4) can be rewritten

$$(5) \quad u_R(S) = \left[1 + R(\theta)\, e^{ikh \cos \theta} \right] \frac{e^{ik\rho}}{\sqrt{\rho}} + O(\rho^{-3/2})$$

and if we compare (5) with (1) we see that

$$(6) \qquad g(\theta) = 1 + R(\theta)\, e^{ikh \cos \theta}$$

Assume for the moment that the boundary condition (1.4) satisfies reciprocity. Then the same argument would yield

$$(7) \qquad g'(\theta) = 1 + R'(\theta)\, e^{ikh \cos \theta}$$

as the angular variation of the far-field if we replace the dielectric interface at $y = 0$ by the boundary condition (1.4). In (7) $R'(\theta)$ is the approximate reflection coefficient defined by (1.4). Consequently if the boundary condition (1.9) satisfies reciprocity we can compare (6) and (7) and make the assertion: Replacement of a dielectric interface by its approximation (1.4) creates an error, in the far-field of a line source, which is proportional to the error in matching the reflection coefficient for plane waves at real angles of incidence.

We shall prove the required reciprocity theorem which validates the above argument.

Proof of Reciprocity

Define two solutions of the wave equation u_R and u_S

$$(8) \qquad (\nabla^2 + k^2)u_R = \delta(R)$$

$$(9) \qquad (\nabla^2 + k^2)u_S = \delta(S)$$

which both satisfy the general boundary condition at $y = 0$

$$(10) \qquad \left(\frac{\partial}{\partial y} + \alpha + \beta \frac{\partial^2}{\partial x^2}\right) u = 0$$

and the radiation condition at infinity

$$(11) \qquad \lim_{\rho \to \infty} \rho^{\frac{1}{2}} \left(\frac{\partial u}{\partial \rho} - iku\right) = 0 , \qquad 0 \leq \theta \leq \pi$$

corresponding to a suppressed time factor $e^{-i\omega t}$. R and S are shorthand for the coordinates of the receiver (r_x, r_y) and source (s_x, s_y)

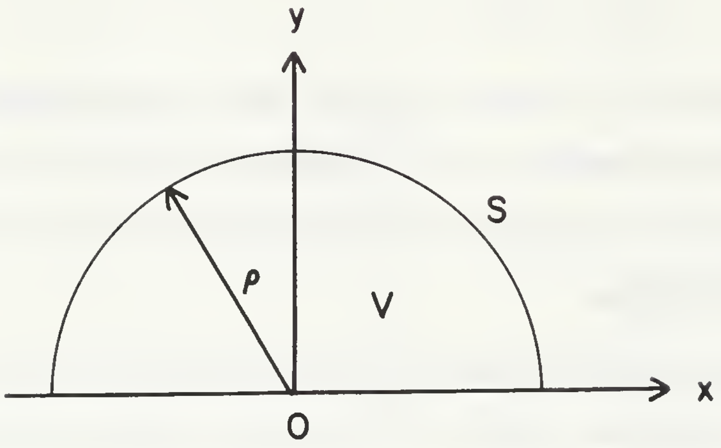
$$\delta(R) = \delta(x - r_x)\,\delta(y - r_y)$$

$$\delta(S') = \delta(x - s_x)\,\delta(y - s_y)$$

Apply Green's theorem to u_R and u_S

$$\int_V (u_R \nabla^2 u_S - u_S \nabla^2 u_R) d\tau = \int_S \left(u_R \frac{\partial u_S}{\partial n} - u_S \frac{\partial u_R}{\partial n}\right) d\sigma$$

where V is the area bounded by the large semi-circle and S is its boundary.



As the radius ρ of the semi-circle goes to infinity we obtain (Appendix C)

$$u_R(S) - u_S(R) = -\beta \int_{-\infty}^{\infty} (u_R \frac{\partial^2 u_S}{\partial x^2} - u_S \frac{\partial^2 u_R}{\partial x^2})\, dx$$

since the integration over the arc vanishes by virtue of the radiation condition. The remaining integrand is a perfect differential

$$\frac{\partial}{\partial x} (u_R \frac{\partial u_S}{\partial x} - u_S \frac{\partial u_R}{\partial x})$$

so this integral vanishes too by the radiation condition thus proving reciprocity i.e.

$$u_R(S) = u_S(R) \quad .$$

As shown in the previous section, one is then justified in stating that approximation of the reflection coefficient for real angles of incidence corresponds to approximating the far-field of a line source and vice versa.

We have therefore shown that the introduction of the new class of boundary conditions instead of the actual matching conditions at a dielectric interface guarantee a small percentage error over the entire far-field. The chief virtue of this replacement is that it permits an explicit approximate solution to otherwise unsolvable problems. Consequently to gain assurance in the use of the boundary condition it would be worthwhile to compare the result of its use as with the exact solution of a non-trivial problem. Such a comparison is discussed in Section 5.

5. Comparison of the Result of the New Method, with an Exact Solution for a Problem Involving Diffraction

A. Exactly Formulated Problem and Solution

We consider the following two-dimensional Wiener-Hopf problem for which the far-field in the positive x-direction can be obtained exactly without need to perform an explicit factorization[2]. A plane wave is incident at grazing incidence along a dielectric interface (cf. Figure 7). For $x \leq 0$ there is a semi-infinite metallic screen at the interface. If

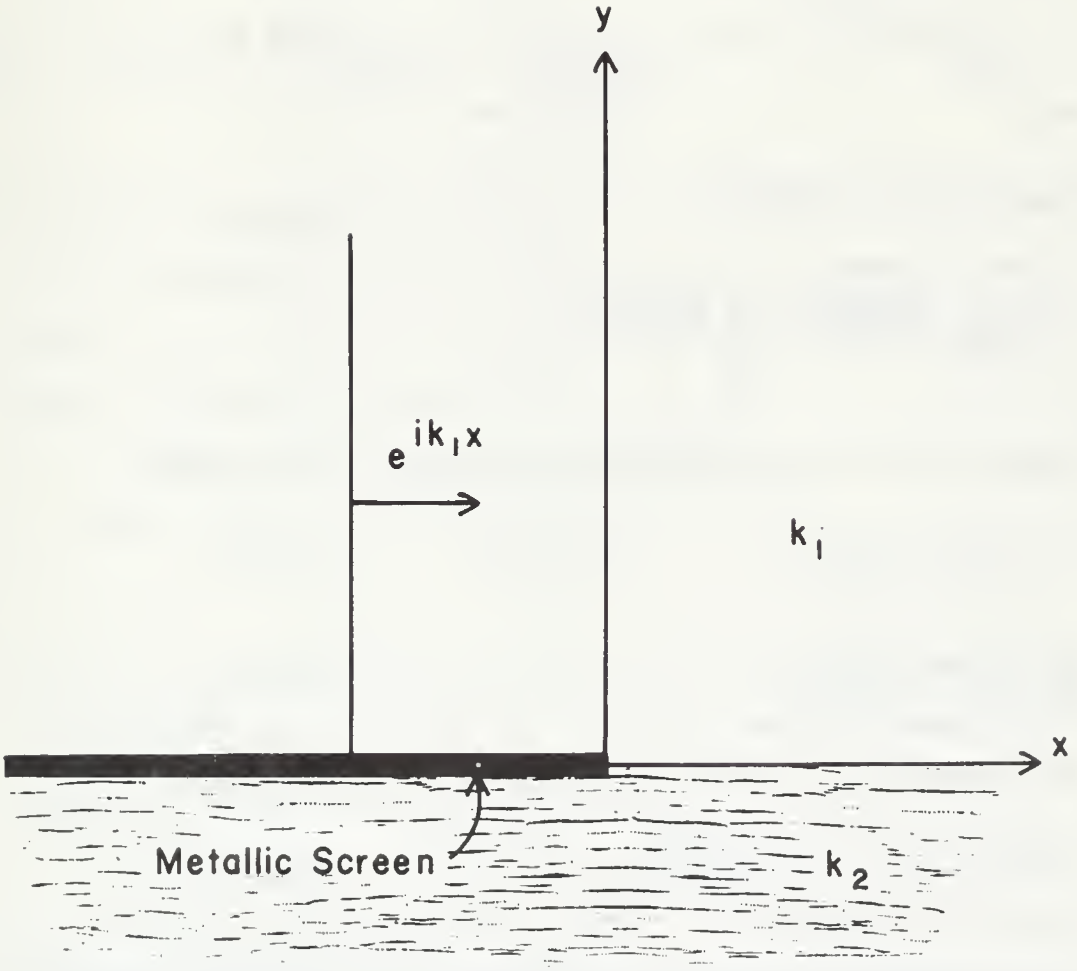


Figure 7: A plane wave launched on a semi-infinite metallic plane at the interface of two dielectrics.

we associate the field $u(x,y)$ with the magnetic component H_z then the electromagnetic boundary conditions to be satisfied are

(1) $$\frac{\partial u}{\partial y} = 0 \qquad x \leq 0, \quad y = 0$$

and

$$\left.\begin{aligned} &(2) \quad u(x,0+) = u(x,0-) \\ &(3) \quad \frac{1}{k_1^2}\,\frac{\partial u(x,0+)}{\partial y} = \frac{1}{k_2^2}\,\frac{\partial u(x,0-)}{\partial y} \end{aligned}\right\} \quad \begin{aligned} &x > 0 \\ &y = 0 \end{aligned}$$

where $u(x,y)$ is the solution of the appropriate wave equation

(4) $$(\nabla^2 + k_1^2)u = 0 \qquad y > 0$$

(5) $$(\nabla^2 + k_2^2)u = 0 \qquad y < 0 .$$

The incident field is taken to be

$$e^{ik_1x} \qquad y > 0 \quad .$$

The boundary condition (1) expresses the vanishing of tangential E at a perfect conductor. Equation (2) guarantees the continuity of H_z and equation (3) implies the continuity of tangential E at a dielectric as can be seen from the Maxwell equation

(4) $$\text{curl } \vec{H} = -i\omega\varepsilon \vec{E}$$

where we assume $H_x = H_y = 0$.

To solve the problem we note that u(x,y) for $y \geq 0$ can be expressed in the form[3]

$$(5)\qquad u(x,y) = e^{ik_1x} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{A^-(\nu)}{\sqrt{k_1^2-\nu^2}}\, e^{i\nu x + i\sqrt{k_1^2 - \nu^2}\, y}\, d\nu$$

where $A^-(\nu)$ is an unknown function. The branch-cuts in the ν-plane are drawn as indicated in Figure 8 . Note that we have assumed a small imaginary part for k_1 and k_2 for purposes of convenience.* In a like manner we can express the solution for $y \leq 0$ as

$$(6)\qquad u(x,y) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{B^-(\nu)\, e^{i\nu x - i\sqrt{k_2^2 - \nu^2}\, y}}{\sqrt{k_2^2 - \nu^2}}\, d\nu$$

where $B^-(\nu)$ is an unknown function of ν; note the sign of the radicals in the exponentials of (4) and (5). They have been so chosen as to assure the outgoing nature of the solution.

By their construction the representations (5) and (6) satisfy the respective wave equations (3) and (4). In addition, if $A^-(\nu)$ and $B^-(\nu)$ are analytic functions of ν for $\text{Im}\,\nu \leq 0$ and have the following behavior in the lower half-plane

$$\text{Im}\,\nu \leq 0 \quad \begin{cases} |A^-(\nu)| \backsim |\nu|^{-\varepsilon_1} \quad \text{as} \quad |\nu| \to \infty , & \varepsilon_1 > 0 \\ |B^-(\nu)| \backsim |\nu|^{-\varepsilon_2} \quad \text{as} \quad |\nu| \to \infty , & \varepsilon_2 > 0 \end{cases}$$

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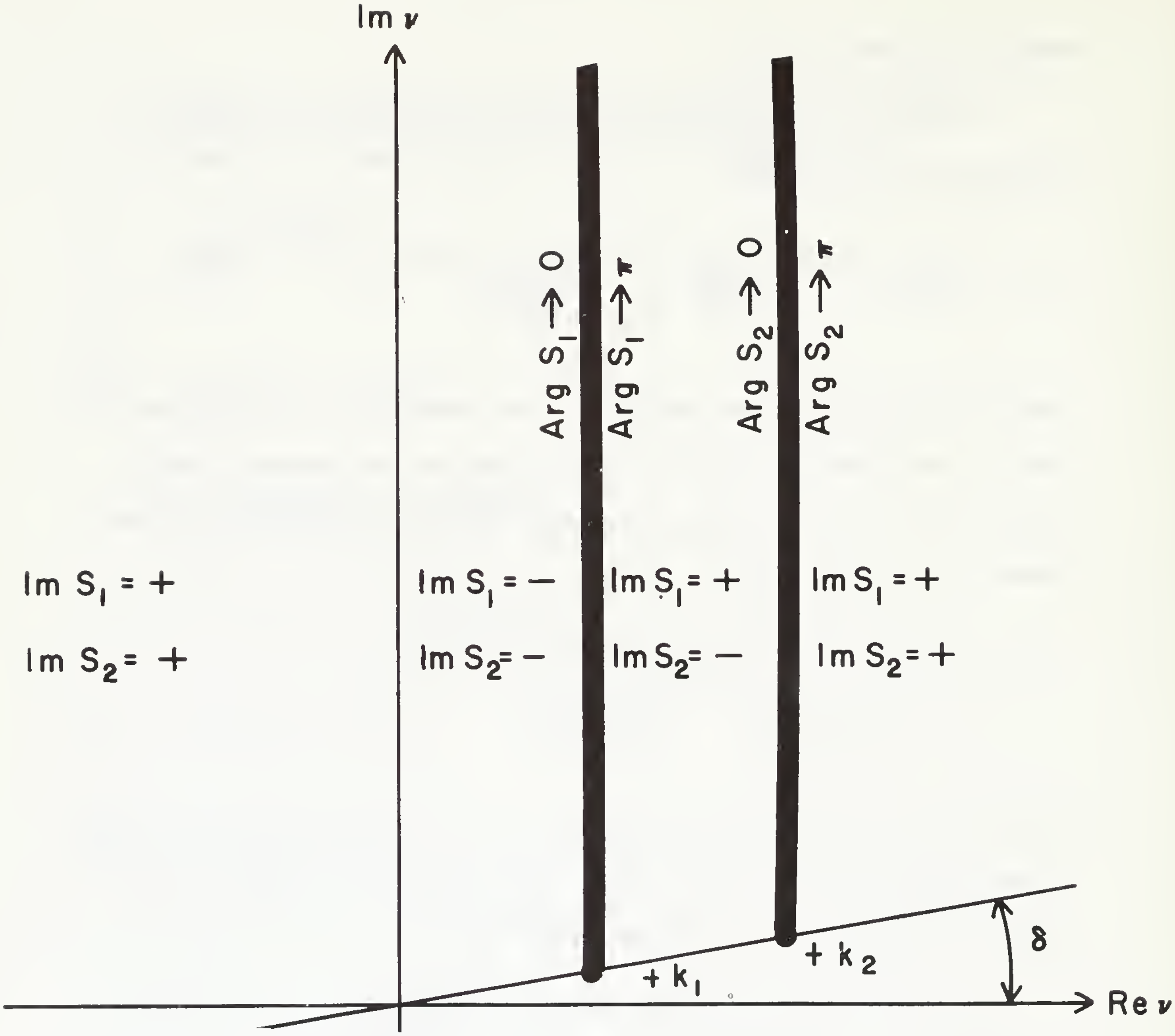
*However the ratio k_1/k_2 is real; in the final result these imaginary parts of k_1 and k_2 will be neglected.



$$S_1(\nu) = \sqrt{k_1^2 - \nu^2}, \quad S_1(0) \equiv +k_1$$

$$S_2(\nu) = \sqrt{k_2^2 - \nu^2}, \quad S_2(0) \equiv +k_2$$

Figure 8: The complex ν-plane; the imaginary parts of $S_1(\nu)$ and $S_2(\nu)$ are greater than or equal to zero on the real ν-axis. Two cuts in the third quadrant are omitted in this figure, for convenience.

then we can close the respective contours for $x \leq 0$, and the boundary condition (1) will be satisfied by Jordan's lemma.

To satisfy the boundary condition (3) we note that it is sufficient to choose

$$(7) \qquad \frac{A^-(\nu)}{k_1^2} = -\frac{B^-(\nu)}{k_2^2}$$

If $G^+(\nu)$ defined

$$(8) \quad \frac{1}{\nu - k_1} + \frac{A^-(\nu)}{\sqrt{k_1^2 - \nu^2}} - \frac{B^-(\nu)}{\sqrt{k_2^2 - \nu^2}} = G^+(\nu)$$

by (8) is an analytic function of ν for $\text{Im}\,\nu \geq 0$ of algebraic order $-\varepsilon_3 < 0$ at infinity then the boundary condition (2) will also be satisfied by Jordan's lemma.

It can be shown [Appendix A] that it is possible to express

$$(9) \quad P^+(\nu)\, P^+(-\nu) = \frac{1}{\sqrt{k_1^2 - \nu^2}} + \frac{k_2^2}{k_1^2\sqrt{k_2^2 - \nu^2}} = \sigma(\nu)$$

where $P^+(\nu)$ is an analytic function regular and zeroless, and $O(\nu^{-1/2})$, for $\text{Im}(\nu + k_1) \geq 0$. We can then rewrite (8) as

$$(10) \qquad \frac{G^+(\nu)}{P^+(\nu)} + \frac{1}{\nu - m}\left[\frac{1}{P^+(m)} - \frac{1}{P^+(\nu)}\right] = \frac{1}{\nu - m}\,\frac{}{P(m)} + A^-(\nu)\, P^+(-\nu)\ .$$

Owing to the assumed behavior of $G^+(\nu)$ and $A^-(\nu)$ at infinity it follows that each side of (10) vanishes at infinity. In addition the left hand side is analytic for $\mathrm{Im}(\nu + k_1) \geq 0$ and the right hand side for $\mathrm{Im}(\nu - k_1) \leq 0$. It follows that each side is the analytic continuation of the other. If we invoke the Liouville theorem we find that each side must vanish separately and hence

$$(11) \qquad A^-(\nu) = -\frac{1}{\nu - k} \frac{P^+(\nu)}{P(k_1)\sigma(\nu)} .$$

If we examine the growth* of $\sigma(\nu)$ and $P^+(\nu)$ we find

$$\lim_{|\nu| \to \infty} A^-(\nu) = O(\nu^{-1/2}),$$

so that for $y = 0$, both u_s and the normal derivatives

$$(12) \qquad \frac{\partial u_s}{\partial y} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} A^-(\nu)\, e^{-i\nu x}\, d\nu$$

are integrable in a vicinity of the origin. For $y = 0$ the diffracted field can be displayed explicitly as

$$(13) \qquad u_s(x,0) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\sqrt{k_1^2 - \nu^2}} \frac{P^+(\nu) e^{i\nu x} d\nu}{P^+(k_1)\sigma(\nu)(\nu - k_1)} .$$

To obtain the field for $y=0$ and $x \gg 1$ we note that since $\sigma(\nu)$ is zeroless (cf. Appendix B) we can deform the contour in (12) to C^* as indicated in Figure 9. The path C^* is compared of two portions: C_1, an integration about the branch cut at k_1, and C_2, another integration about the branch cut at k_2.

---

*cf. Equation (9) of the present section, and the last equation of Appendix A.

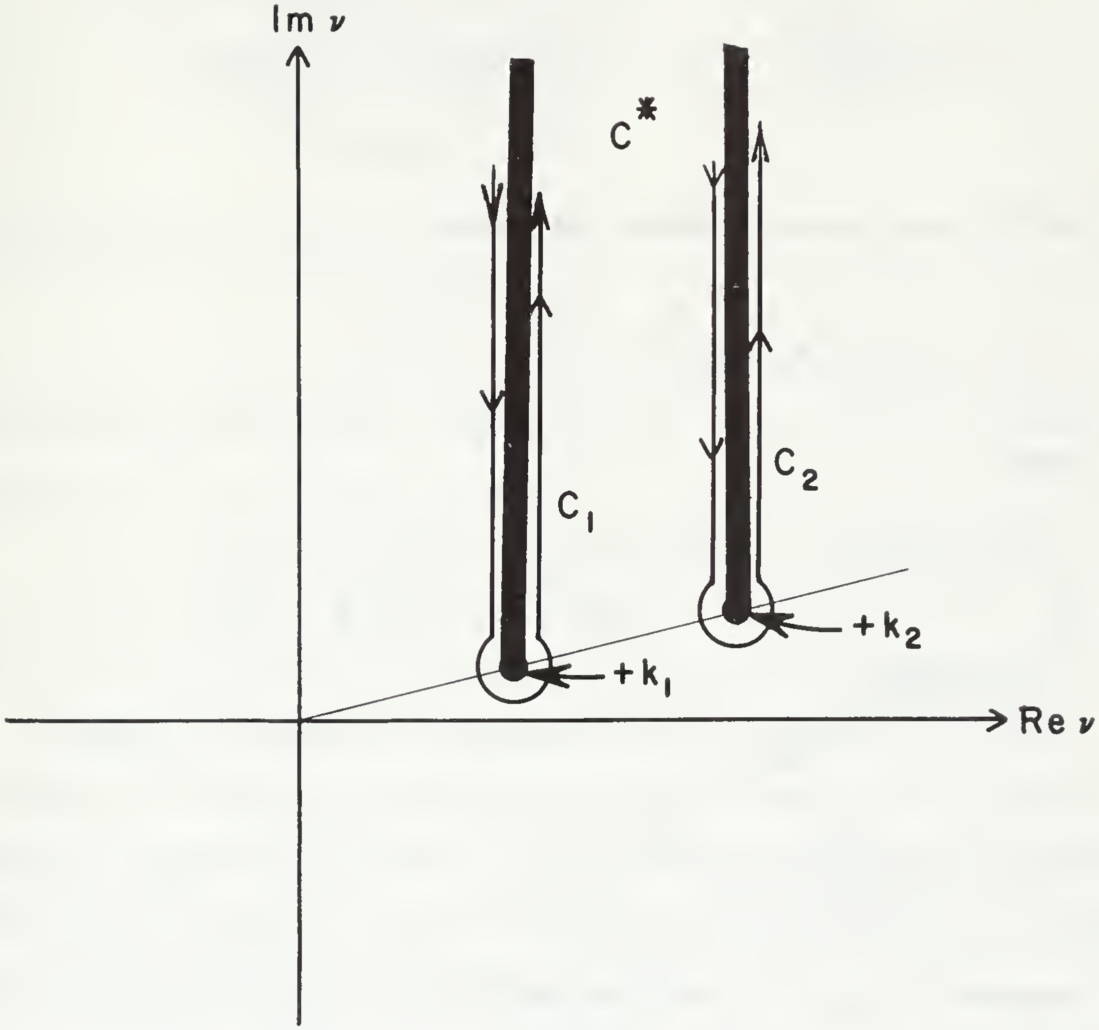


Figure 9: The contour C^* is composed of both C_1 and C_2.

For $x \gg 1$ we can obtain an asymptotic evaluation of (12) by the procedure of Van der Waerden[7] if we discuss each integral separately. Consider first the integral about C_1: Let $s = \sqrt{\nu - k_1}$, then

$$I(\nu) = \frac{P^{+}(\nu)}{\sqrt{k_1^2 - \nu^2}\,\sigma(\nu)} = P^{+}(\nu)\left[\frac{k_1^2\sqrt{k_2^2-\nu^2}}{k_1^2\sqrt{k_2^2-\nu^2} + k_2^2\sqrt{k_1^2-\nu^2}}\right]$$

admits a power series expansion in s about k_1

$$I(s) = P(k_1)\left[a_o + a_1 s\right] + O(s^2)$$

where

$$a_o = I(k_1) = 1$$

and

$$a_1 = \frac{dI(s)}{ds}\Bigg|_{s=k_1}$$

$$= -\left(\frac{k_2}{k_1}\right)^2\left(k_2^2 - k_1^2\right)^{-1/2}\sqrt{2k_1}\,.$$

Consequently the entire integrand of (12) is

$$-\frac{1}{2\pi i}\left[\frac{1}{\nu - k} + \frac{a_1}{\sqrt{\nu - k}}\right]e^{i\nu x} + O\left[(\nu-k)^{+\frac{1}{2}}\right]$$

and following Van der Waerden we can integrate term by term to obtain

$$(14)\qquad u_s(x,0) = -e^{ik_1x}\left[1 + \frac{a_1\Gamma\left(\frac{1}{2}\right)}{\pi i\sqrt{x}\,e^{i\pi/4}}\right] + O\left[x^{-\frac{3}{2}}\right]$$

as the contribution of the integration about C_1 with the terms $P(k_1)$ cancelling each other.

The leading term in an expansion about k_2 is proportional to

$$\sqrt{\nu - k_2}$$

which contributes an amount

$$O(x^{-3/2})$$

to the asymptotic development. Consequently the entire field for $y = 0$, $x \gg 1$ is

$$\boxed{(15) \quad u(x,0) = -\left(\frac{k_2}{k_1}\right)^2 \sqrt{\frac{2k_1}{k_2^2 - k_1^2}} \; \frac{e^{ik_1 x - \pi i/4}}{\sqrt{\pi x}} + O(x^{-3/2}) \quad .}$$

Equation (15) gives the complete field for $y=0$ and $k_1 x \gg 1$ for the problem of a plane wave incident along a semi-infinite metallic screen located at a dielectric interface. In part B we shall obtain a corresponding result when the dielectric interface is replaced by our new boundary condition. It will be seen that the results compare very well.

B. Approximately Reformulated Problem and Solution.

In part A, formula (15) represents the forward scattering at a dielectric interface of a plane wave launched at grazing incidence on a semi-infinite metallic screen. We can obtain a corresponding expression to (15) if we replace the dielectric interface by our new boundary condition. The solutions for the two problems can be compared in the forward direction ($x \gg 1$, $y=0$) to test the utility of the boundary condition.

The problem posed in this part will be identical with that of part A with the exception that the boundary condition

$$(16) \qquad \left(\frac{1}{ik_1} \frac{\partial}{\partial y} + A + \frac{B}{k_1^2} \frac{\partial^2}{\partial x^2} \right) u(x,y) = 0$$

will replace the dielectric interface at $x > 0$, $y=0$.

A and B are the parameters introduced in Section 1. In that section we indicated that there existed some flexibility in the choice of their values. Consequently we shall leave them unspecified except for the restriction that they be real and satisfy

$$A > B$$

[cf. (1.8)] which will imply the lack of zeros of the expression*

$$\frac{1}{k_1} \sqrt{k_1^2 - \nu^2} + A - \frac{B\nu^2}{k_1^2}$$

Once again we introduce the construction

$$(17) \qquad u(x,y) = e^{ik_1 x} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{A^-(\nu)}{\sqrt{k_1^2 - \nu^2}} \, e^{i\nu x + i\sqrt{k_1^2 - \nu^2}\, y} \, d\nu$$

which will satisfy the boundary condition (1) if $A^-(\nu)$ is analytic for $\operatorname{Im} \nu \leq 0$ and,

$$(17a) \qquad A^-(\nu) \backsim |\nu|^{-\varepsilon} \quad \text{as } |\nu| \to \infty \qquad \operatorname{Im} \nu \leq 0$$

---

*See Appendix C.

for some $\varepsilon \geq 0$. For convenience define*

$$(18) \qquad \sigma_A(\nu) = \frac{\frac{1}{k_1}\sqrt{k_1^2-\nu^2} + A - \frac{B\nu^2}{k_1^2}}{k_1^2 - \nu^2}$$

which can be factorized by the well-known procedure of Appendix A

$$(19) \qquad \sigma_A(\nu) = P_A^+(\nu)\, P_A^+(-\nu)$$

where $P^+(\nu)$ is an analytic function for $\mathrm{Im}(\nu+k) > 0$ such that $|P^+(\nu)|$ is bounded above and below by positive constants in its half plane of regularity. It follows that if $G^+(\nu)$ defined by

$$(20) \qquad G^+(\nu) = \frac{A-B}{\nu-k_1} + \sqrt{k_1^2-\nu^2}\, A^-(\nu)\, \sigma_A(\nu)$$

is analytic and behaves like

$$G^+(\nu) \sim \nu^{-\varepsilon},\ |\nu| \to 0\ , \qquad\qquad \mathrm{Im}\,\nu > 0$$

in the upper half plane the boundary condition (16) will be satisfied(by Jordan's lemma)

As usual, we shall now solve for $A^-(\nu)$ and $G^+(\nu)$ by appeal to Liouville's theorem. Rewrite (20) as

$$\frac{G^+(\nu)}{P_A^+(\nu)\,\sqrt{k_1+\nu}} + \frac{A-B}{\nu-k_1}\left[\frac{1}{\sqrt{2k_1}P_A^+(k_1)} - \frac{1}{\sqrt{k_1+\nu}\,P_A^+(\nu)}\right] =$$

$$(21) \qquad \frac{A-B}{(\nu-k_1)\sqrt{2k_1}\,P_A(k_1)} + A^-(\nu)\sqrt{k_1-\nu}\,P_A^+(-\nu) \quad .$$

---

*At this stage let k_1 have a small positive imaginary part which will be made to vanish later.

Owing to the factor $\sqrt{k_1-\nu}$ of $A^-(\nu)$ in (21) we cannot immediately conclude that each side of (21) vanishes. However we have another condition at our disposal: The behavior of the solution in the vicinity of the origin. We shall seek the solution such that the highest order derivative $\frac{\partial^2 u}{\partial x^2}$ which appears in the boundary condition (16) be integrable. For this demand it is sufficient that[*]

$$|A^-(\nu)| \backsim \nu^{-\delta} \qquad \text{as} \quad |\nu| \to \infty \ , \quad \text{Im}\ \nu \geq 0 \ ,$$

where $\delta > 1$. A solution of this type will be obtained thus verifying the assumption. As a consequence it follows that each side of (21) is analytic in a half plane, shares a common strip of analyticity and vanishes at infinity in its respective half plane. Invoke the Liouville theorem and obtain

$$A^-(\nu) = \frac{-(A-B)}{(\nu-k_1)\sqrt{2k_1}\ P_A(k_1)} \Big/ \sqrt{k_1-\nu} \quad P_A^+(-\nu)$$

or alternately by $\big((18)$ and $(19)\big)$,

$$A^-(\nu) = -\frac{(A-B)\ P_A^+(\nu)}{(\nu-k_1)^{3/2}\sigma_A(\nu)\sqrt{2k_1}\ P_A(k_1)} \ .$$

The integral representation of the scattered field becomes

$$\text{(22)} \qquad u_s(x,y) = -\frac{A-B}{2\pi i\sqrt{2k_1}\ P_A(k_1)} \int_{-\infty}^{\infty} \frac{\sqrt{k_1+\nu}\ P_A^+(\nu)e^{i\nu x + i\sqrt{k^2-\nu^2}\ y}}{(\nu-k_1)\left[\frac{1}{k_1}\sqrt{k_1^2-\nu^2} + A - \frac{B}{k_1^2}\nu^2\right]} d\nu$$

---

[*] Alternatively we could have required $\varepsilon > 1$ in connection with the equation (17a).

We can evaluate the far field for $x \gg 1$, and $y = 0$ in the same fashion as Part A. We obtain

$$u_s(x,0) = \frac{-(A-B)}{2\pi i\sqrt{2k_1}\,P_A(k_1)} \int_{C_1} \frac{P_A^+(\nu)\, e^{i\nu x}\, d\nu}{(\nu-k_1)\left[\frac{\sqrt{k_1^2-\nu^2}}{k_1} + A - \frac{B\nu^2}{k_1^2}\right]}$$

(23)

$$= -e^{ik_1x}\left[1 + \frac{\sqrt{2k_1}}{A-B}\,\frac{e^{-\pi i/4}}{\sqrt{\pi x}}\right] + O(x^{-3/2})$$

and the total field $e^{ik_1x} + u_s(x,0)$ is

$$\boxed{(24) \qquad u(x,0) = -\frac{\sqrt{2k_1}}{A-B}\,\frac{e^{ik_1x - \pi i/4}}{k_1\sqrt{\pi x}} + O(x^{-3/2})}$$

for $y=0$, $x \gg 1$.

Note that for both the true and the approximate problems the leading term in the scattered field cancelled the incident field.* This is of course a conseqence of the fact that the reflection coefficient for grazing incidence is -1 for both problems. It is more important to notice that for both problems if $x \gg 1$, the functional form of the leading terms (15) and (24) agree, and differ only by a constant factor

$$\frac{k_2^2}{k_1^2}\,\frac{1}{\sqrt{k_2^2-k_1^2}} \leftrightarrows \frac{1}{k_1(A-B)}$$

---

*cf. equation (15) of this section.

or, since $A_{TM} = 1/n$, regardless of which of the various sets of (A,B) we use,

$$\frac{n}{k_1\sqrt{1-(\frac{1}{n})^2}} \rightleftarrows \frac{1}{k_1(\frac{1}{n}-B)} \quad .$$

Consequently the percentage error between these two coefficients is

$$(25) \qquad \text{P. E.} = \left[1 - \frac{\sqrt{1-(\frac{1}{n})^2}}{1 - n\,B_{TM}} \right] \cdot 100 \quad .$$

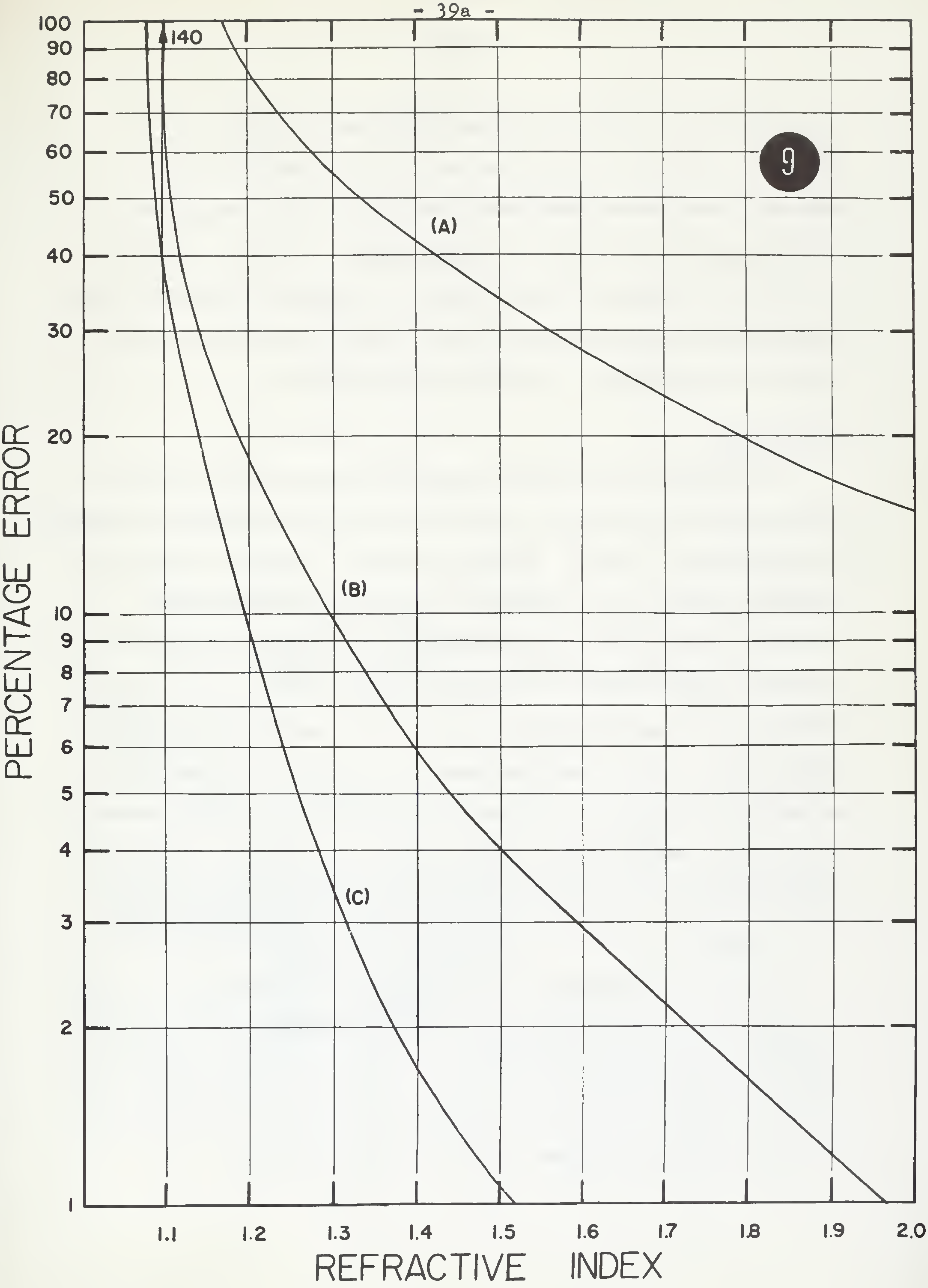
Since B_{TM} is of the order $O(1/n^3)$ [cf. (1.9)] for fitting at any real angle we see that the error decreases with increasing n.

In line with the remarks in Section 1 we may observe that if in the far-field $x >> 1$, $y=0$, the most important direction had been defined as grazing incidence then one would have chosen B so as to match the derivative of the reflection coefficient at grazing incidence i.e.

$$(26) \qquad B_{TM} = \frac{1}{n}\left[1 - \sqrt{1-(\frac{1}{n})^2} \right]$$

then there would have been no error at all in (24) as compared with (15).

As we remarked earlier if we compare (24) with (15) we notice that the functional form of the leading terms are the same but with differing constants for various choices of B. We can compute the percentage error between these leading coefficients. In Graph 9 we plot three curves:


9
140
(A)
(B)
(C)
PERCENTAGE ERROR
100
90
80
70
60
50
40
30
20
10
9
8
7
6
5
4
3
2
1
REFRACTIVE INDEX
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0

(A) represents the "normal impedance" approximation $\frac{1}{ik}\frac{\partial u}{\partial y} + Au = 0$ obtained by setting $B = 0$ (a related condition customarily used to represent almost perfect conductors); (B) shows the error obtained by a binomial expansion of the radical given by (1.11); (C) is the error if B is chosen as indicated by Graph 3 which is a matching at Brewster's angle. Notice that the big improvement is from (A) to (B), an improvement which arises from the additional term in the boundary condition.

6. Conclusion

To conclude Part I it may be urged that mixed boundary conditions with more than two terms have a great deal to offer in approximation techniques for boundary-value problems. We have introduced a technique by which the transition conditions at an interface between two dielectrics can be conveniently and accurately replaced by such a single mixed boundary condition. The method is useful when one is interested in the field in only one of the dielectrics. We have tested the accuracy of the results so obtainable, in several ways. First: we have demonstrated that this is a reasonable and well-defined approach by proving the uniqueness and reciprocity theorems for a half space. Secondly: when a single interface is present, so that there is no diffraction, it has been proved that the maximum percentage error, over the entire far field of an arbitrarily placed line source is small.

Additional confidence in the technique has been demonstrated by using the boundary condition to replace the transition conditions in a non-trivial problem which does involve diffraction. A comparison of the results of the approximate version and the exactly formulated problem has shown that there is excellent agreement between the two in a certain respect.

Appendix A

The function

$$\sigma(\nu) = \frac{1}{\sqrt{k_1^2 - \nu^2}} + \frac{k_2^2}{k_1^2 \sqrt{k_2^2 - \nu^2}}$$

can be factorized as

$$\sigma(\nu) = P^+(\nu)\ P^+(-\nu)$$

where $P^+(\nu)$ is analytic and zeroless for $\operatorname{Im} \nu \geq \operatorname{Im} -k_1$ by an application of the Cauchy integral theorem. Define

$$S(\nu) \equiv \frac{k_1^2 \sqrt{k_2^2 - \nu^2} + k_2^2 \sqrt{k_1^2 - \nu^2}}{(k_1^2 + k_2^2) \sqrt{k_2^2 - \nu^2}}$$

then

$$\sigma(\nu) = \frac{k_1^2 + k_2^2}{\sqrt{k_1^2 - \nu^2}} \cdot S(\nu) .$$

In appendix B it is shown that $S(\nu)$ and hence $\sigma(\nu)$ is zeroless in the cut ν-plane, consequently

$$\ln S(\nu)$$

is an analytic function for

$$|\operatorname{Im} \nu| < \operatorname{Im} k_1$$

and square integrable where $|k_1| < |k_2|$. By the Cauchy integral theorem

(1) $\ln S(\nu) = \ln p(\nu) - \ln n(\nu)$

where

$$\ln p(\nu) = \frac{1}{2\pi i} \int_{-\infty - i\beta}^{+\infty - i\beta} \frac{1}{\nu - \zeta} \ln S(\zeta)\, d\zeta$$

and

$$\ln n(\nu) = \frac{1}{2\pi i} \int_{-\infty + i\beta}^{+\infty + i\beta} \frac{1}{\nu - \zeta} \ln S(\zeta)\, d\zeta$$

where β is any real number $0 < \beta < |k_1|$. By virtue of the evenness of $S(\nu)$ we have the relation

$$p(\nu) = \frac{1}{n(-\nu)}$$

and by definition $p(\nu)$ is analytic for $\text{Im}\ \nu \geq \beta$. Thus

$$S(\nu) = p(\nu)\ p(-\nu)$$

and if we define

$$P^{+}(\nu) = \frac{\sqrt{k_1^2 + k_2^2}\ p(\nu)}{\sqrt{k_1 + \nu}}$$

we have the desired factorization

$$\sigma(\nu) = P^{+}(\nu)\ P^{+}(-\nu) \quad .$$

Since $p(\nu)$ is $O(1)$ as $|\nu| \longrightarrow \infty$ it follows that

$$P^{+}(\nu) = O(\nu^{-1/2})$$

as $|\nu| \longrightarrow \infty$.

Appendix B

The expression

$$(1) \quad R(\nu) = k_1^2 \sqrt{k_2^2 - \nu^2} + k_2^2 \sqrt{k_1^2 - \nu^2}$$

can have zeros only in a region where the imaginary parts of the radicals have different signs. In the upper half-plane $\text{Im}\ \nu \geq 0$ this can only arise in the half strip $\text{Im}\ \nu \geq 0$,

$$\text{Re}\ k_1 \leq \text{Re}\ \nu \leq \text{Re}\ k_2.$$

If we multiply (1) by $R^*(\nu)$, where

$$(2) \qquad R^*(\nu) \equiv k_1^2\sqrt{k_2^2 - \nu^2} - k_2^2\sqrt{k_1^2 - \nu^2}$$

we find that the roots of $R(\nu)\cdot R^*(\nu)$ arise for

$$\nu_o^2 = \frac{k_1^2 k_2^2}{k_1^2 + k_2^2}$$

$$= k_1^2 \cdot \frac{1}{1 + (\frac{k_1}{k_2})^2}$$

so that ν_o lies on that portion of the ray $\theta = \delta$ between k_1 and the origin. But on this segment a root of $R(\nu)$ is impossible, since it is to the left of both cuts. Since $R(\nu)$ is even there are also no zeros in the lower ν-plane.

Appendix C

We can show that the expression

$$Z(\nu) = \frac{1}{k_1^2}\sqrt{k_1^2 - \nu^2} + (A_{TM} - B_{TM}\frac{\nu^2}{k_1^2})$$

lacks zeros in the ν-plane if we recall (1.7) and (1.11)

$$A_{TM} = \frac{k_1}{k_2} = \frac{1}{n} < 1$$

$$B_{TM} = A_{TM}\left[1 - \sqrt{1 - \frac{1}{n^2+1}}\right]$$

We have allowed k_1 and k_2 to be complex, but in such a manner that the ratio k_1/k_2 is real; this in turn implies that A_{TM} and B_{TM} are real. Furthermore, it is clear that they are positive.

Define the expression $Z^*(\nu)$

$$Z^*(\nu) = \frac{1}{k_1^2} \sqrt{k_1^2 - \nu^2} - (A_{TM} - B_{TM} \frac{\nu^2}{k_1^2})$$

and we shall show that the zeros of the product $Z(\nu) \cdot Z^*(\nu)$ are necessarily those of $Z^*(\nu)$. For this purpose it is convenient to introduce the new variable $\xi = \nu/k_1$. We can calculate that the product

$$Z(\nu) \cdot Z^*(\nu) = Z(k_1\xi) \cdot Z^*(k_1\xi) = 1 - \xi^2 - (A_{TM} - B_{TM}\, \xi^2)^2$$

is a quartic in ξ and hence has four zeros. It is easily shown that these four roots lie on the real and imaginary ξ-axes. To verify this point note that since $A_{TM} < 1$ the expression ZZ^* is positive at the origin. As we move along either coordinate axis in any direction we find that at infinity ZZ^* has changed sign and is negative. This implies the existence of two roots on each of the ξ-coordinate axes. We have $A_{TM} > B_{TM}$. Hence, by our choice of the positive branch of the radical the expression

$$Z(k_1\xi) = \sqrt{1 - \xi^2} + (A_{TM} - B_{TM}\, \xi^2)$$

has no roots on the segment $|\mathrm{Re}\ \xi| \leq 1$. Of course there are also no roots on the segment $|\mathrm{Re}\ \xi| > 1$, since the radical is imaginary there. Hence Z cannot have a zero on the real ξ-axis. Since $-\xi^2$ is positive on the imaginary axis a root of Z cannot exist there either. Hence all the zeros of ZZ^* must be those of Z^* and we have proved the assertion: $Z(\nu)$ is zeroless in the cut ν-plane.

Appendix D

(For notation, refer to
Section 4)

Apply the wave equation to

$$\int_V (u_R \nabla^2 u_S - u_S \nabla^2 u_R) \, d\tau$$

and we obtain by (3.1) and (3.2)

$$\int_V \left[u_R\ \delta(S) - u_S\ \delta(R) \right] d\tau \quad .$$

Recall the fundamental property of the δ-function.

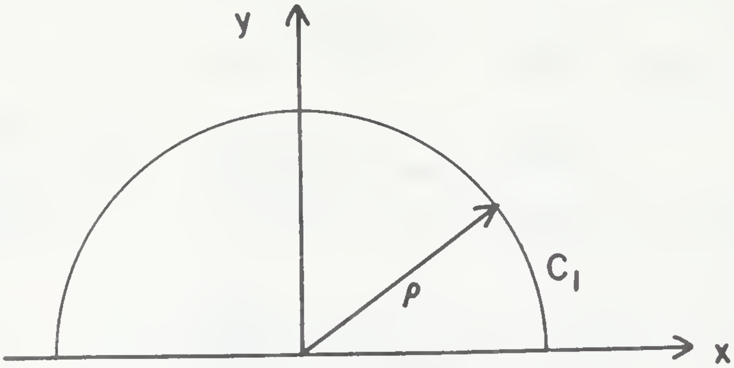
$$\int_R g(x)\ \delta(x - \xi)\ dx = g\ (\xi)$$

for any range of integration R including ξ. Hence as $\rho \rightarrow \infty$ the volume integral becomes $u_R(S) - u_S(R)$.

The integral about S, the boundary of V, is

$$\int_S \left(u_R \frac{\partial u_S}{\partial n} - u_S \frac{\partial u_R}{\partial n}\right)\ d\sigma \quad .$$

where S is a large semicircle with the x-axis as diameter



and C_1 is the circular portion of S; we calculate that

$$\lim_{\rho \rightarrow \infty} \int_S d\sigma = \int_{-\infty}^{\infty} dx \quad + \lim_{\rho \rightarrow \infty} \int_{C_1} d\sigma \quad .$$

Consider the integral about C_1 separately, if we assume that we can interchange the limit and integration

$$\lim_{\rho\to\infty} \int_{C_1} = \int_{C_1} \lim_{\rho\to\infty} \left(u_R \frac{\partial u_S}{\partial \rho} - u_S \frac{\partial u_R}{\partial \rho} \right) d\sigma$$

by (3.4)

$$= ik \int_{C_1} (u_R\, u_S - u_S\, u_R)\, d\sigma$$

$$= 0 \quad .$$

For the integral about the x-axis apply the boundary condition (3.3).

$$\int_{-\infty}^{\infty} (u_R \frac{\partial u_S}{\partial y} - u_S \frac{\partial u_R}{\partial y})\, dx$$

$$= - \int_{-\infty}^{\infty} \left[u_R\, (\alpha u_S + \beta \frac{\partial^2 u_S}{\partial x^2}) - u_S(\alpha u_R + \beta \frac{\partial^2 u_R}{\partial x^2}) \right] dx$$

$$= -\beta \int_{-\infty}^{\infty} (u_R \frac{\partial^2 u_S}{\partial x^2} - u_S \frac{\partial^2 u_R}{\partial x^2})\, dx \quad .$$

to obtain the desired result.

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